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Chapter 8
Poisson's Equation

8.1 General solution

When free charges exist in space, the electric potential $\varphi(\vec{r})$ satisfies Poison's equation, which is written as

$$\nabla^2\varphi(\vec{r}) = -\frac{\rho_f(\vec{r})}{\varepsilon\varepsilon_0}. \tag{3.15}$$

Under Dirichlet boundary conditions, where the potential on a surface S is specified as $\varphi(\vec{r}_s) = f(\vec{r}_s)$ or Neumann boundary conditions, where $\frac{\partial\varphi}{\partial n} = \hat{n}\cdot\nabla\varphi$, the solution to Poisson's equation is unique. Specifically, we have a unique solution $\varphi_1(\vec{r})$,

$$\nabla^2\varphi_1(\vec{r}) = -\frac{\rho_f(\vec{r})}{\varepsilon\varepsilon_0}. \tag{8.1}$$

From this, the general solution to Poisson's equation can be expressed as,

$$\varphi_P(\vec{r}) = \varphi_1(\vec{r}) + \varphi_L(\vec{r}), \tag{8.2}$$

where $\varphi_L(\vec{r})$ satisfies the Laplace equation, $\nabla^2\varphi_L(\vec{r}) = 0$. Using the principle of superposition, the total potential satisfies

$$\nabla^2\varphi_P(\vec{r}) = \nabla^2\varphi_1(\vec{r}) + \nabla^2\varphi_L(\vec{r}) = -\frac{\rho_f(\vec{r})}{\varepsilon\varepsilon_0}. \tag{8.3}$$

Below, we illustrate the general solution with two examples: a point charge in a conducting cavity and a point charge near a grounded conducting plane.

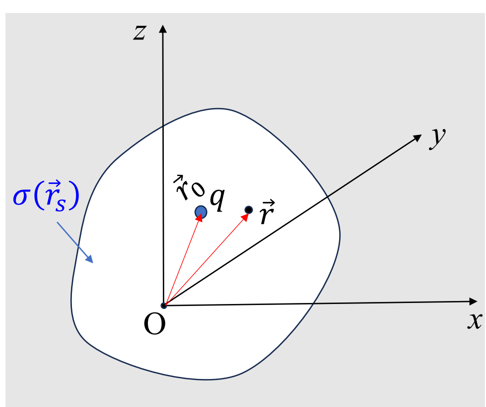

Fig. 8.1 A point charge inside a metal cavity.

For a point charge q, the Poisson's equation can be written as,

$$\nabla^2\varphi(\vec{r}) = -\frac{q\delta(\vec{r}-\vec{r}_0)}{\varepsilon_0}. \quad (8.4)$$

A particular solution to this equation is the potential generated by the point charge,

$$\varphi_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}_0|}. \quad (8.5)$$

If the point charge is placed within a cavity inside a conductor (see **Figure 8.1**), the boundary condition requires that the potential on the cavity surface remains constant ($\varphi(\vec{r}_s) = constant$). To satisfy this, an induced surface charge distribution $\sigma(\vec{r}_s)$ arises on the cavity walls, contributing an additional potential. The total potential is then,

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}_0|} + \frac{1}{4\pi\varepsilon_0}\oiint_S \frac{\sigma(\vec{r}_s)}{|\vec{r}-\vec{r}_s|}dS', \quad (8.6)$$

Here, the second term satisfies the Laplace equation and represents the potential due to the induced surface charge distribution.

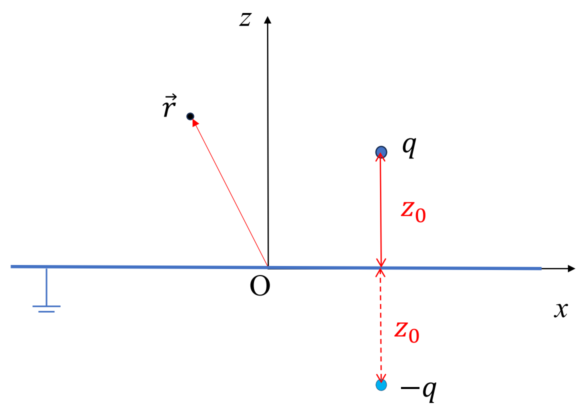

Fig. 8.2 The image charge of a point charge on top of an infinitely large grounded conducting plate.

The Method of Images (MoI) offers a powerful way to solve Poisson's equation in the presence of conducting boundaries. Consider a point charge q located at $(0, 0, z_0)$ above a grounded, infinitely large conducting plane (**Figure 8.2**). To satisfy the boundary condition that the potential $\varphi(\rho, z = 0) = 0$ on the plane $(z = 0)$, we introduce an image charge $-q$ at $(0, 0, -z_0)$. The potential in the upper half-space $(z \geq 0)$ at $\vec{r}$ is then given by the superposition of the potentials from the real charge and the image charge,

$$\varphi(\rho, z \geq 0) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{\rho^2+(z-z_0)^2}} - \frac{1}{\sqrt{\rho^2+(z+z_0)^2}}\right]. \quad (8.7)$$

In this equation, the first term $\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\rho^2+(z-z_0)^2}}$ corresponds to the potential of the real charge and satisfies Poisson's equation, while the second term $-\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\rho^2+(z+z_0)^2}}$ corresponds to the potential of the image charge and satisfies the Laplace equation, mimicking the potential due to the surface charge distribution on the conducting plane.

The force between the point charge and the conducting plane is attractive and can be calculated as,

$$\vec{F} = -\frac{q^2}{16\pi\varepsilon_0 z_0^2}\hat{z}. \tag{8.8}$$

The induced surface charge density on the plane is,

$$\sigma = -\varepsilon_0 \frac{\partial\varphi}{\partial z}\bigg|_{z=0}. \tag{8.9}$$

8.2 Method of Images

The MoI is a powerful mathematical technique used to solve Poisson's equation when a charge interacts with conducting boundaries. For a point charge, the general solution to Poisson's equation is,

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}_0|} + \varphi_L(\vec{r}). \tag{8.10}$$

Here, $\varphi_L(\vec{r})$ satisfies the Laplace equation and represents the potential due to the conducting boundary. Using the MoI, $\varphi_L(\vec{r})$ can be expressed as the potential of an image charge, which simplifies the solution while maintaining the physical boundary conditions. Below, we examine several practical examples of the MoIs.

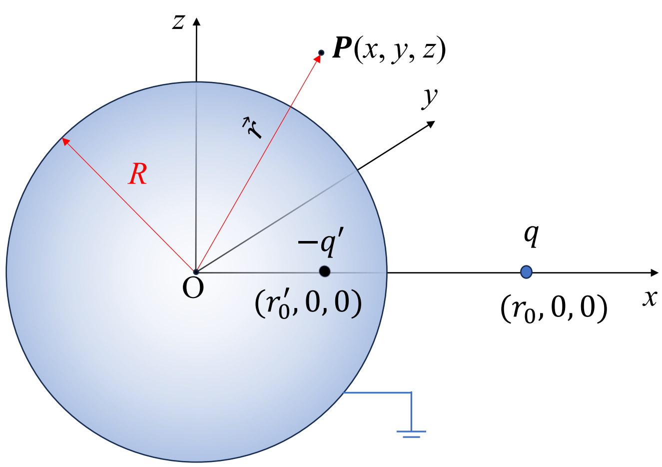

Fig. 8.3 A point charge outside a grounded conductor sphere, with an image charge inside the sphere.

8.2.1 Point charge near a grounded conducting sphere

Consider a point charge q located at $\vec{r}_0$ outside a grounded conductor sphere of radius R as shown in **Figure 8.3**. To ensure that the sphere's surface potential remains zero ($\varphi(r=R)=0$), an image charge q' is introduced at location $\vec{r}_0'$ inside the sphere. The total potential outside the sphere ($r \geq R$) can be written as,

$$\varphi(r \geq R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}_0|} + \frac{1}{4\pi\varepsilon_0}\frac{q'}{|\vec{r}-\vec{r}_0'|}. \tag{8.11}$$

Given the azimuthal symmetry of the system, both $\vec{r}_0$ and $\vec{r}_0'$ lie along the same radial direction $\hat{n}$. At the surface of the sphere, $\vec{r} = R\hat{n}$, the boundary condition requires

$$\varphi(\vec{r} = R\hat{n}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_0 - R} + \frac{1}{4\pi\varepsilon_0}\frac{q'}{R - r_0'}$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r_0}\frac{1}{1 - R/r_0} + \frac{1}{4\pi\varepsilon_0}\frac{q'}{R}\frac{1}{1 - r_0'/R} = 0. \qquad (8.12)$$

One way to make above equation $\varphi = 0$ is,

$$\begin{cases} \frac{q}{r_0} = -\frac{q'}{R} \\ 1 - \frac{R}{r_0} = 1 - r_0'/R \end{cases}. \qquad (8.13)$$

Solving this condition gives the values for the image charge and its position,

$$\begin{cases} q' = -\frac{R}{r_0} q \\ r_0' = \frac{R^2}{r_0} \end{cases}, \qquad (8.14)$$

Thus, the total potential outside the sphere is

$$\varphi(r \geq R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r} - \vec{r}_0|} - \frac{1}{4\pi\varepsilon_0}\frac{Rq/r_0}{|\vec{r} - \vec{r}_0 R^2/r_0^2|}. \qquad (8.15)$$

The induced surface charge density on the conducting sphere can be calculated as

$$\sigma = -\varepsilon_0 \frac{\partial\varphi}{\partial r}\bigg|_{r=R} = -\frac{q}{4\pi R}\frac{r_0^2 - R^2}{\left(r_0^2 + R^2 - 2r_0 R\cos\theta\right)^{3/2}}. \qquad (8.16)$$

Here θ is the angle between $\vec{r}$ and $\vec{r}_0$. The attractive force between the point charge and the sphere is,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}\left(\frac{R}{r_0}\right)^3 \left(1 - \frac{R^2}{r_0^2}\right)^{-2}. \qquad (8.17)$$

Alternative Scenarios for a Point Charge Outside a Conducting Sphere

(1) Conducting sphere not grounded

If the conducting sphere is not grounded, the image charge q' and its position $\vec{r}_0'$ remain the same, as given by **Equation 8.14**. However, since the sphere must remain electrically neutral, a uniform charge distribution of $-q'$ is induced on the sphere's surface to balance q'. This $-q'$ charge should be uniformly distributed on the sphere surface, which induces another potential, $-\frac{1}{4\pi\varepsilon_0}\frac{q'}{|\vec{r}|}$. The total potential at $r \geq R$ can be written as,

$$\varphi(r \geq R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r} - \vec{r}_0|} - \frac{1}{4\pi\varepsilon_0}\frac{\frac{Rq}{r_0}}{\left|\vec{r} - \frac{\vec{r}_0 R^2}{r_0^2}\right|} + \frac{1}{4\pi\varepsilon_0}\frac{\frac{R}{r_0}q}{|\vec{r}|}. \qquad (8.18)$$

(2) Conducting sphere with a net charge Q

If the sphere initially carries a net charge Q, the charge Q will distribute uniformly on the sphere's surface. The total potential outside the sphere is then,

$$\varphi(r \geq R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}_0|} - \frac{1}{4\pi\varepsilon_0}\frac{\frac{Rq}{r_0}}{\left|\vec{r}-\frac{\vec{r}_0 R^2}{r_0^2}\right|} + \frac{1}{4\pi\varepsilon_0}\frac{Q+\frac{R}{r_0}q}{|\vec{r}|}. \tag{8.19}$$

(3) Conducting sphere at a fixed potential V_0

If the conducting sphere is held at a fixed potential V_0, the boundary condition becomes $\varphi(r = R) = V_0$. To satisfy this, an additional potential term is added

$$\varphi(r \geq R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}_0|} - \frac{1}{4\pi\varepsilon_0}\frac{\frac{Rq}{r_0}}{\left|\vec{r}-\frac{\vec{r}_0 R^2}{r_0^2}\right|} + \frac{V_0 R}{|\vec{r}|}. \tag{8.20}$$

In this expression, the first two terms ensure that the potential vanishes at $r = R$, while the third term raises the potential to V_0.

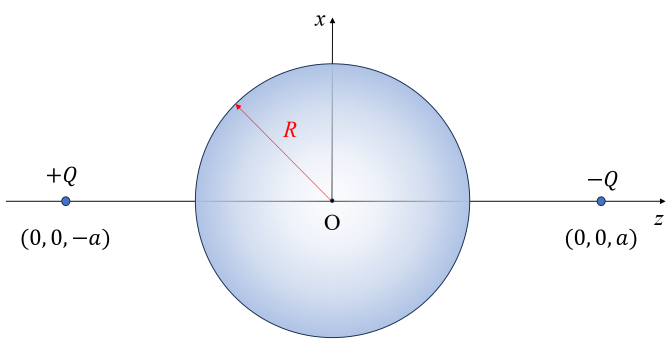

Fig. 8.4 Two point charges outside a conductor sphere.

8.2.2 Method of images for a conducting sphere in a uniform external field

The MoI can be used to determine the potential generated by a uniform external electric field applied to a conducting sphere. As shown in **Figure 8.4**, the uniform external field is simulated by placing two point charges $\pm Q$ symmetrically on opposite sides of the conducting sphere at a distance a. The external field is recovered by taking the limit $a \to \infty$, ensuring that the field approaches uniformity over the sphere. Each point charge induces an image charge inside the conducting sphere. The magnitudes and positions of these image charges are given by

$$\begin{cases} q'_{\pm} = \mp QR/a \\ r'_{\pm} = \mp R^2/a \end{cases}. \tag{8.21}$$

The total potential $\varphi(\vec{r})$ at a point $\vec{r}$ outside the sphere is the superposition of the potentials due to the point charges $\pm Q$ and their corresponding image charges $q'_{\pm}$. The expression for $\varphi(\vec{r})$ is

$$\varphi(\vec{r}) = \frac{Q/4\pi\varepsilon_0}{(r^2+a^2+2ra\cos\theta)^{\frac{1}{2}}} - \frac{Q/4\pi\varepsilon_0}{(r^2+a^2-2ra\cos\theta)^{\frac{1}{2}}} - \frac{RQ/4\pi\varepsilon_0}{a\left(r^2+\frac{R^{4^2}}{a}+\frac{2R^2 r\cos\theta}{a}\right)^{\frac{1}{2}}}$$
$$+\frac{RQ/4\pi\varepsilon_0}{a(r^2+R^4/a^2-2R^2 r\cos\theta/a)^{\frac{1}{2}}}. \tag{8.22}$$

For a uniform external field, we take the limit $a \to \infty$. Expanding **Equation 8.22** using a Taylor series for large a, the potential becomes

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[-\frac{2Q}{a^2} r\cos\theta + \frac{2Q}{a^2}\frac{R^3}{r^2}\cos\theta\right] + \cdots \tag{8.23}$$

The first term in **Equation 8.23** corresponds to the applied uniform field, while the second term accounts for the field distortion caused by the induced charges on the sphere. Comparing this with a uniformly applied external field $\vec{E}_0 = -\nabla\varphi(\vec{r})$, we identify the relationship between Q and the applied field E_0

$$E_0 = \frac{1}{2\pi\varepsilon_0}\frac{Q}{a^2}. \tag{8.24}$$

Substituting this into the expression for $\varphi(\vec{r})$, the potential in the presence of the sphere is

$$\varphi(\vec{r}) = -E_0\left[r - \frac{R^3}{r^2}\right]\cos\theta = -E_0 z\left[1 - \frac{R^3}{r^3}\right]. \tag{8.25}$$

This result shows the distortion of the potential due to the conducting sphere. It matches the solution obtained from solving Laplace's equation directly, as described in **Example 7.2**.

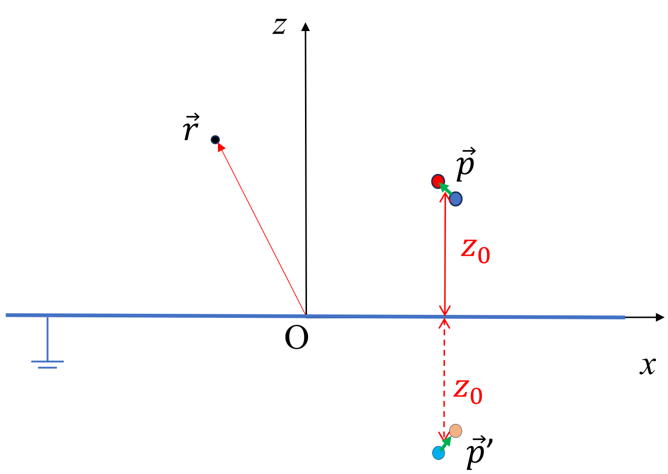

Fig. 8.5 A dipole above a grounded conductor plane.

8.2.3 A dipole near a grounded conductor plane

The MoI can also be applied to solve for the potential generated by a dipole near a grounded conducting plane. In this case, the dipole consists of two charges, $+q$ and $-q$, separated by a small distance d. The corresponding image charges are placed symmetrically below the conducting plane, such that the conducting surface at $z = 0$ remains at zero potential. **Figure 8.5** illustrates this setup, showing the real dipole and its image dipole. Assume the real dipole is in the x-z plane, with its dipole moment $\vec{p}$ making an angle α with the z-axis. The dipole moments of the real and image dipoles are

$$\begin{cases} \vec{p} = p\cos\alpha\,\hat{k} - p\sin\alpha\,\hat{\imath} \\ \vec{p}' = p\cos\alpha\,\hat{k} + p\sin\alpha\,\hat{\imath} \end{cases}. \tag{8.26}$$

The real dipole is located at $(0,0,z_0)$, while the image dipole is located at $(0,0,-z_0)$. The total potential $\varphi(\vec{r})$ at a point $\vec{r} = (x, y, z)$ at $z \geq 0$ is the superposition of the potentials generated by the real dipole and its image dipole. This can be expressed as

$$\begin{aligned} \varphi(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3} + \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}'\cdot(\vec{r}-\vec{r}_0')}{|\vec{r}-\vec{r}_0'|^3} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{-px\sin\alpha + p(z-z_0)\cos\alpha}{[x^2+y^2+(z-z_0)^2]^{3/2}} + \frac{1}{4\pi\varepsilon_0}\frac{px\sin\alpha + p(z+z_0)\cos\alpha}{[x^2+y^2+(z+z_0)^2]^{3/2}}. \end{aligned} \tag{8.27}$$

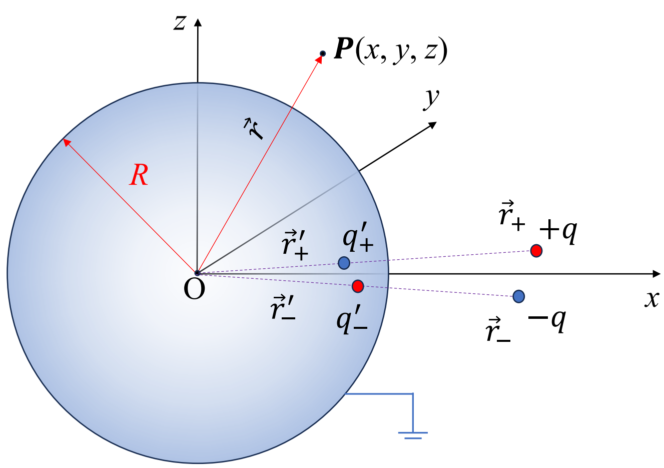

Fig. 8.6 A dipole near a grounded conductor sphere.

8.2.4 A dipole near a conductor sphere

The MoI can also be used to analyze the interaction between a dipole and a conducting sphere. In this scenario, as shown in **Figure 8.6**, the positive charge $+q$ of the dipole is located at $\vec{r}_+$, and the negative charge $-q$ is located at $\vec{r}_-$. The original dipole moment $\vec{p}$ can be written as,

$$\vec{p} = q(\vec{r}_+ - \vec{r}_-). \tag{8.28}$$

Each charge induces an image charge within the sphere to maintain the boundary conditions imposed by the conductor, which is given by

$$\begin{cases} q'_{\pm} = \mp qR/r_{\pm} \\ \vec{r}'_{\pm} = \mp R^2\vec{r}_{\pm}/r_{\pm}^2 \end{cases}. \tag{8.29}$$

These image charges collectively produce an image dipole moment, which is calculated as,

$$\vec{p}' = q'_{+}\vec{r}'_{+} + q'_{-}\vec{r}'_{-} = q\frac{R^3}{r_{-}^3}\vec{r}_{-} - q\frac{R^3}{r_{+}^3}\vec{r}_{+}. \tag{8.30}$$

To simplify the expressions, we define $\vec{r}_0 = \frac{\vec{r}_{+}+\vec{r}_{-}}{2}$, $r_{\pm} = \vec{r}_0 \pm \frac{\vec{r}_{+}-\vec{r}_{-}}{2}$, and thus

$$\frac{1}{r_{\pm}^3} = \frac{1}{\left|\vec{r}_0 \pm \frac{\vec{r}_{+}-\vec{r}_{-}}{2}\right|^3} = \frac{1}{\left[r_0^2 + \frac{|\vec{r}_{+}-\vec{r}_{-}|^2}{4} \pm \vec{r}_0\cdot(\vec{r}_{+}-\vec{r}_{-})\right]^{3/2}} \approx \frac{1}{r_0^3}\left[1 \mp \frac{3}{2}\frac{\vec{r}_0\cdot(\vec{r}_{+}-\vec{r}_{-})}{r_0^2}\right]. \tag{8.31}$$

Here we assume that $|\vec{r}_{+} - \vec{r}_{-}| \ll |\vec{r}_0|$ for simplification. Substituting **Equation 8.31** into **Equation 8.30**, the image dipole moment becomes

$$\begin{aligned} \vec{p}' &= \frac{R^3}{r_0^2}\left[q(\vec{r}_{-} - \vec{r}_{+}) + \frac{3}{2}\frac{\vec{r}_0\cdot q(\vec{r}_{+}-\vec{r}_{-})}{r_0^2}(\vec{r}_{+} - \vec{r}_{-})\right] \\ &= \frac{R^3}{r_0^2}\left[-\vec{p} + \frac{3}{2}\frac{\vec{r}_0\cdot\vec{p}}{r_0^2}(\vec{r}_{+} - \vec{r}_{-})\right]. \end{aligned} \tag{8.33}$$

Since the induced image charges are not symmetrically distributed, an excess charge can accumulate on the conducting sphere, which can be calculated as,

$$q_{excess} = Rq\left(\frac{1}{r_{-}} - \frac{1}{r_{+}}\right) = Rq\frac{r_{+}-r_{-}}{r_0^2}. \tag{8.34}$$

For small separations,

$$r_{+} - r_{-} \approx -d\cos\alpha, \tag{8.35}$$

with $d = |\vec{r}_{+} - \vec{r}_{-}|$ is the dipole length and α is the angle between $\vec{r}_0$ and $\vec{p}$. Substituting this approximation gives,

$$q_{excess} \approx Rq\frac{d\cos\alpha}{r_0^2} = R\frac{\vec{p}\cdot\vec{r}_0}{r_0^3}. \tag{8.36}$$

Therefore, the potential generated outside the sphere can be written as,

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{p}\cdot(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3} + \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}'\cdot(\vec{r}-\vec{r}'_0)}{|\vec{r}-\vec{r}'_0|^3}\right] - \frac{1}{4\pi\varepsilon_0}\frac{q_{excess}}{|\vec{r}|}. \tag{8.37}$$

Here $\vec{r}'_0 = \frac{\vec{r}'_{+}+\vec{r}'_{-}}{2}$ represents the center of the image dipole.

8.2.5 A charged wire near a conductive cylinder

Consider an infinitely long wire carrying a uniform linear charge density λ, positioned parallel to an infinitely long grounded conductive cylinder with radius R. The goal is to find the electric potential $\varphi(x, y)$ at a point $\boldsymbol{P}(x, y)$ outside the cylinder. This problem is inherently two-dimensional because, due to the cylindrical symmetry, $\varphi(\vec{r})$ depends only on the x- and y-coordinates.

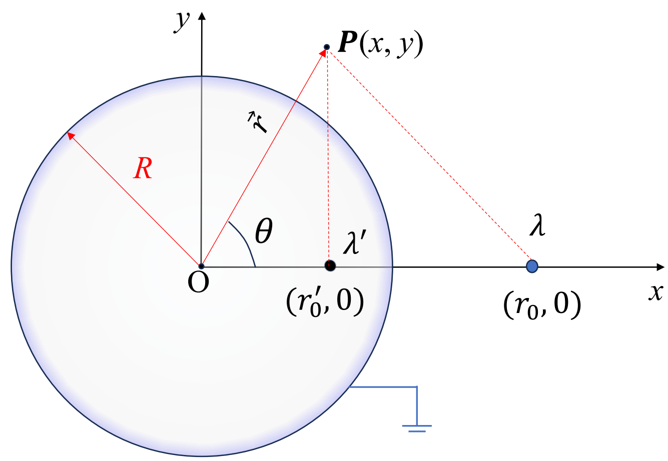

Fig. 8.7 A charged wire near a grounded conducting cylinder.

Let the charged wire be positioned at a distance r_0 from the center of the cylinder, the potential due to the wire alone, without the presence of the cylinder, is given by,

$$\varphi(\vec{r}) = -\frac{\lambda}{2\pi\varepsilon_0}\ln|\vec{r}-\vec{r}_0| = -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+r_0^2-2rr_0\cos\theta}. \tag{8.38}$$

To satisfy the boundary condition imposed by the grounded conducting cylinder (where the potential on its surface is zero), we introduce an an image charge. This image charge is represented by a fictitious wire of linear charge density λ' located at a position $\vec{r}_0'$ inside the cylinder. The potential due to both the original charge and the image charge wire is,

$$\varphi(\vec{r}) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+r_0^2-2rr_0\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\sqrt{r^2+r_0'^2-2rr_0'\cos\theta} = -\frac{1}{2\pi\varepsilon_0}\ln\frac{(r^2+r_0^2-2rr_0\cos\theta)^{\lambda/2}}{(r^2+r_0'^2-2rr_0'\cos\theta)^{-\lambda'/2}}. \tag{8.39}$$

The grounded cylinder imposes the condition that $\varphi(\vec{r})$ →constant on its surface ($r = R$). This leads to the relationship

$$\lambda = -\lambda'. \tag{8.40}$$

Also the tangential electric field $E_\theta(r,\theta)$ can be written as,

$$E_\theta(r,\theta) = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{r_0\sin\theta}{r^2+r_0^2-2rr_0\cos\theta} - \frac{r_0'\sin\theta}{r^2+r_0'^2-2rr_0'\cos\theta}\right]. \tag{8.41}$$

When $r \to R$, $E_\theta(r,\theta) = 0$. Therefore,

$$r_0'[R^2+r_0^2] = r_0[R^2+r_0'^2]. \tag{8.42}$$

There are two solutions for r_0': $r_0' = r_0$ and $r_0' = R^2/r_0$. We take the second solution, because the image charge must lie inside the cylinder. Therefore, the potential becomes,

$$\varphi(\vec{r}) = -\frac{\lambda}{4\pi\varepsilon_0}\ln\frac{r^2+r_0^2-2rr_0\cos\theta}{r^2+R^4/r_0^2-2rR^2\cos\theta/r_0}. \quad (8.43)$$

On the cylinder surface ($r = R$), the potential simplifies to

$$\varphi(R) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r_0}{R}. \quad (8.44)$$

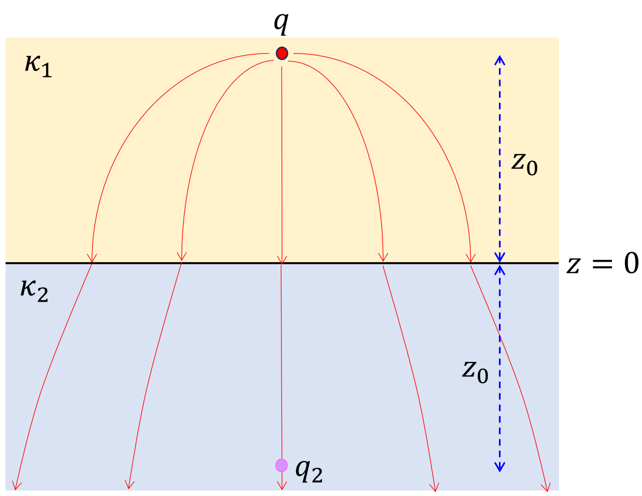

Fig. 8.8 The electric field lines of a point charge at the boundary between materials κ_1 and κ_2.

8.2.6 Method of image at a dielectric boundary

When a point charge is placed near an interface between two infinitely large dielectric materials with dielectric constants κ_1 and κ_2, the problem requires the use of the method of images to determine the electric potential and fields in the regions separated by the interface at $z = 0$. The configuration is shown in **Figure 8.8**. At the interface ($z = 0$), the boundary conditions are derived from the continuity of the potential and the displacement field

$$\begin{cases} \varphi_1|_{z=0} = \varphi_2|_{z=0} \\ \kappa_1 \left.\frac{\partial\varphi_1}{\partial z}\right|_{z=0} = \kappa_2 \left.\frac{\partial\varphi_2}{\partial z}\right|_{z=0} \end{cases}, \quad (8.45)$$

Here φ_1 and φ_2 are the electric potentials in regions $z > 0$ and $z < 0$, respectively. To solve for the electric potential $\varphi(\vec{r})$, we divide the configuration into two regions: $z > 0$ and $z < 0$.

1. Region $z > 0$

The electric field in this region results from the original point charge q and the field induced by the interface. The interface can be treated as if it contains a fictitious image charge q_2 located at a distance z_0 below the interface (in $z < 0$). The potential in this region is given by

$$\varphi_1(\vec{r}) = \frac{1}{4\pi\kappa_1\varepsilon_0}\left[\frac{q}{\sqrt{\rho^2+(z-z_0)^2}} + \frac{q_2}{\sqrt{\rho^2+(z+z_0)^2}}\right], \quad (8.46)$$

Where $\rho = \sqrt{x^2 + y^2}$ is the radial distance in the *x-y* plane.

2. Region $z < 0$

In this region, the electric field can be treated as originating from an image charge q_1, located at distance z_0 above the interface (in $z < 0$). The potential in this region is,

$$\varphi_2(\vec{r}) = \frac{1}{4\pi\kappa_2\varepsilon_0}\frac{q_1}{\sqrt{\rho^2+(z-z_0)^2}}. \tag{8.47}$$

Apply the boundary conditions, we have

$$\begin{cases} \frac{1}{4\pi\kappa_1\varepsilon_0}\left[\frac{q}{\sqrt{\rho^2+z_0^2}} + \frac{q_2}{\sqrt{\rho^2+z_0^2}}\right] = \frac{1}{4\pi\kappa_2\varepsilon_0}\frac{q_1}{\sqrt{\rho^2+z_0^2}} \\ \frac{1}{4\pi\varepsilon_0}\frac{-qz_0}{\sqrt{\rho^2+z_0^2}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2z_0}{\sqrt{\rho^2+z_0^2}} = \frac{1}{4\pi\varepsilon_0}\frac{-q_1z_0}{\sqrt{\rho^2+z_0^2}} \end{cases}.$$

Solving these equations gives,

$$\begin{cases} q_1 = \frac{2\kappa_2}{\kappa_1+\kappa_2}q \\ q_2 = \frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}q \end{cases}. \tag{8.48}$$

The interaction between the real point charge q and its image charge q_2 leads to a force. This force is directed perpendicular to the interface and has a magnitude,

$$\vec{F} = -\frac{1}{4\pi\kappa_1\varepsilon_0}\frac{qq_2}{(2z_0)^2}\hat{z}. \tag{8.49}$$

The presence of the dielectric materials induces polarization surface charges at the interface. These surface charge densities are given by

$$\begin{cases} \sigma_P^1 = -\varepsilon_0(\kappa_1 - 1)\left.\frac{\partial\varphi_1}{\partial z}\right|_{z=0} = \frac{qz_0}{2\pi(\rho^2+z_0^2)^{3/2}}\frac{\kappa_2}{\kappa_1}\frac{\kappa_1-1}{\kappa_1+\kappa_2},\ z \geq 0 \\ \sigma_P^2 = -\varepsilon_0(\kappa_2 - 1)\left.\frac{\partial\varphi_2}{\partial z}\right|_{z=0} = \frac{-qz_0}{2\pi(\rho^2+z_0^2)^{3/2}}\frac{\kappa_2-1}{\kappa_1+\kappa_2},\ z \leq 0 \end{cases}. \tag{8.50}$$

Cylindrical Dielectric Interface with a Uniformly Charged Line

Consider a dielectric cylinder with dielectric constant κ_1 embedded in a surrounding medium with dielectric constant κ_2. Inside the cylinder lies a uniformly charged line with a linear charge density λ. The interface between the cylinder and the surrounding medium is at radius R. This setup divides the problem into two regions: inside the dielectric cylinder ($r < R$) and outside the cylinder in the surrounding medium ($r \geq R$).

The goal is to determine the electric potential in both regions using the method of images and by applying boundary conditions.

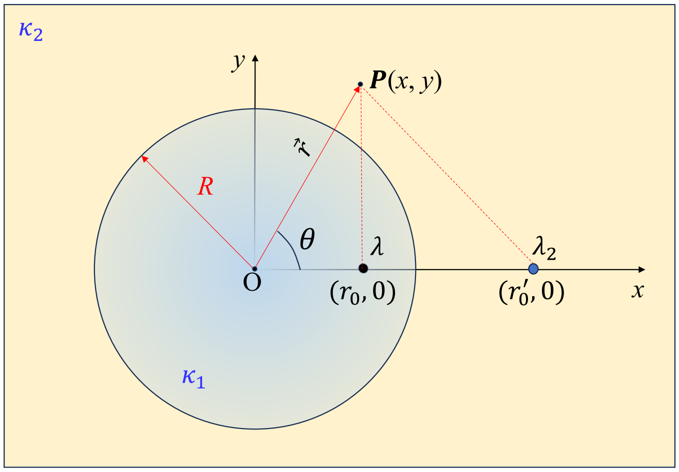

Fig. 8.9 The dielectric cylinder (with dielectric constant κ_1) embedded in a different dielectric medium with dielectric constant κ_2, with a uniformly charged line.

<u>Region 1: $r \geq R$ (outside the cylinder)</u>

In the region outside the cylinder ($r \geq R$), the field is influenced by the original line charge λ and an image charge λ_2 located within the cylinder at a distance R^2/r_0 from the cylinder's center. Here, r_0 is the radial position of the original line charge. The potential in this region is

$$\varphi_2(\vec{r}) = -\frac{\lambda}{2\pi\kappa_2\varepsilon_0}\ln\sqrt{(x-r_0)^2+y^2} - \frac{\lambda_2}{2\pi\kappa_2\varepsilon_0}\ln\sqrt{(x-R^2/r_0)^2+y^2}. \quad (8.51)$$

<u>Region 2: $r < R$ (inside the cylinder)</u>

Inside the cylinder ($r < R$), the field is influenced by the original line charge λ, and an image charge λ_1 located at the same position as λ. However, there is an additional net charge density $\lambda - \lambda_1$, which contributes to the potential. The total potential in this region is

$$\varphi_1(\vec{r}) = -\frac{\lambda_1}{2\pi\kappa_1\varepsilon_0}\ln\sqrt{(x-r_0)^2+y^2} - \frac{\lambda-\lambda_1}{2\pi\kappa_1\varepsilon_0}\ln\sqrt{x^2+y^2}. \quad (8.52)$$

Using the boundary conditions, by matching **Equations 8.51** and **8.52**, we have,

$$\begin{cases} \lambda_2 = \frac{\kappa_2-\kappa_1}{\kappa_1+\kappa_2}\lambda \\ \lambda_1 = \frac{2\kappa_1}{\kappa_1+\kappa_2}\lambda \end{cases}. \quad (8.53)$$

8.3 Green's Function Method

8.3.1 General solution

Green's function provides a powerful framework to solve Poisson's equation under specific boundary conditions. It connects the field generated by a point charge source to the solution of the Poisson equation and is sometimes referred to as a source function or impact function. Green's function represents the potential or field generated by a point charge. For Poisson's equation

$$\nabla^2\varphi(\vec{r}) = -\frac{\rho_f(\vec{r})}{\varepsilon\varepsilon_0}, \tag{3.15}$$

the corresponding Green's function is defined by,

$$\nabla^2 G(\vec{r},\vec{r}') = -\frac{\delta(\vec{r}-\vec{r}')}{\varepsilon_0}, \tag{8.54}$$

where $\delta(\vec{r}-\vec{r}')$ represents a unit point charge located at $\vec{r}'$, and $G(\vec{r},\vec{r}')$ is the potential generated by this point charge. In free space, the Green's function is derived directly from Coulomb's law,

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}\prime|}. \tag{8.55}$$

For an arbitrary charge distribution $\rho(\vec{r}')$, the potential in free space can then be expressed as,

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_{V\prime}\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}\prime|}dV' = \iiint_{V\prime} G(\vec{r},\vec{r}')\rho(\vec{r}')dV'. \tag{3.13}$$

In general, $G(\vec{r},\vec{r}')$ represents the potential generated by a unit point charge under specific boundary conditions. For example, consider the potential of a charge distribution near an infinite conducting plane. Using the method of images, the Green's function in this scenario is

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{[\rho^2+\rho\prime^2-2\rho\rho\prime\cos(\phi-\phi\prime)+(z-z\prime)^2]^{1/2}} - \frac{1}{[\rho^2+\rho\prime^2-2\rho\rho\prime\cos(\phi-\phi\prime)+(z+z\prime)^2]^{1/2}}\right\}. \tag{8.55}$$

If the Green's function for a specific boundary condition is known, as shown in **Figure 8.10A**, Poisson's equation (**Equation 3.15**) can be solved using **Green's second identity**,

$$\iiint dV'\left(f\nabla'^2 g - g\nabla'^2 f\right) = \oiint_S dS'\hat{n}'\cdot(f\nabla' g - g\nabla' f). \tag{8.56}$$

Let $f(\vec{r}') = \varphi(\vec{r}')$ and $g(\vec{r}') = G(\vec{r},\vec{r}')$. Substituting these into the identity gives

$$\iiint dV'\left[\varphi(\vec{r}')\nabla'^2 G(\vec{r},\vec{r}') - G(\vec{r},\vec{r}')\nabla'^2\varphi(\vec{r}')\right]$$

$$= \oiint_S dS'\hat{n}'\cdot[\varphi(\vec{r}')\nabla' G(\vec{r},\vec{r}') - G(\vec{r},\vec{r}')\nabla'\varphi(\vec{r}')]$$

According to **Equation 8.54**, we have

$$\varphi(\vec{r}) = \iiint dV' G(\vec{r},\vec{r}')\rho(\vec{r}') - \varepsilon_0\oiint_S dS'\varphi(\vec{r}')\frac{\partial G(\vec{r},\vec{r}\prime)}{\partial n'} + \varepsilon_0\oiint_S dS' G(\vec{r},\vec{r}')\frac{\partial\varphi(\vec{r}\prime)}{\partial n'}. \tag{8.56}$$

This expression contains: 1. A bulk integral, $\iiint dV' G(\vec{r},\vec{r}')\rho(\vec{r}')$, representing the potential generated by the source charges; 2. Two surface integrations, one ensures that the solution satisfies the boundary conditions, and the other accounts for the potential generated by induced charge distributions at the boundaries, i.e., satisfy the Laplace equation.

For Dirichlet boundary condition, $\varphi(\vec{r})|_{\vec{r}_s} = f$, the Green's function satisfies $G(\vec{r},\vec{r}')|_{\vec{r}_s} = 0$. Substituting this condition into **Equation 8.56** simplifies the potential,

$$\varphi(\vec{r}) = \iiint dV' G(\vec{r},\vec{r}')\rho(\vec{r}') - \varepsilon_0 \oiint_S dS' \varphi(\vec{r}') \frac{\partial G(\vec{r},\vec{r}')}{\partial n'}. \tag{8.57}$$

For Neumann boundary condition, $\left.\frac{\partial\varphi}{\partial n}\right|_{\vec{r}_s} = f(\vec{r})$, the Green's function must satisfy a specific normalization. A typical choice is $\left.\frac{\partial\varphi}{\partial n}\right|_{\vec{r}_s} = -\frac{1}{\varepsilon_0 A}$, where A is the total surface area. This gives the potential,

$$\varphi(\vec{r}) = \langle\varphi\rangle_s + \iiint dV' G(\vec{r},\vec{r}')\rho(\vec{r}') + \varepsilon_0 \oiint_S dS' G(\vec{r},\vec{r}') \frac{\partial\varphi(\vec{r}')}{\partial n'}, \tag{8.58}$$

where $\langle\varphi\rangle_s = \frac{1}{A}\oiint_S dS'\varphi(\vec{r}')$, which is the average potential at the boundary. The Neumann boundary problem usually does not occur in electrostatics.

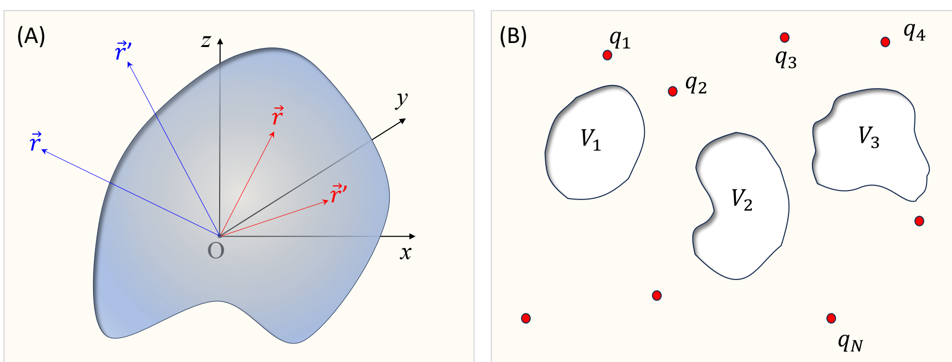

Fig. 8.10 (A) Inside and outside a finite volume for Green's functions. (B) A distributed charge and volume system.

8.3.2 Dirichlet Green's Function

The reciprocal property of Green's functions is given by,

$$G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r}). \tag{8.59}$$

For a finite volume, two Green's functions are typically required, $G_{in}(\vec{r},\vec{r}')$ for the inside of the volume and $G_{ex}(\vec{r},\vec{r}')$ for the outside.

For a charge distribution with boundaries (**Figure 8.10B**), the potential is,

$$\varphi(\vec{r}) = \iiint dV' G(\vec{r},\vec{r}')\rho(\vec{r}') + \varepsilon_0 \sum_k \oiint_{S_k} dS'_k V_k \frac{\partial G(\vec{r},\vec{r}')}{\partial n'_k}. \tag{8.60}$$

8.3.3 Method of Finding the Green's Function

There are two methods to find Green's function, one is the Method of image. This is applicable when symmetry simplifies the placement of image charges to satisfy boundary conditions. The second method is the eigenfunction expansion. The eigenfunctions of the Laplace operator satisfy,

$$\nabla^2\psi_n(\vec{r}) = -\lambda_n\psi_n(\vec{r}), \tag{8.61}$$

with the boundary conditions $\psi_n(\vec{r}_s) = 0$. The eigen value λ_n are real and positive, and the normalized eigenfunctions should be complete, i.e.,

$$\sum_n \psi_n(\vec{r})\psi_n^*(\vec{r}') = \delta(\vec{r} - \vec{r}'). \tag{8.62}$$

Thus the Green's function can be written as,

$$G_D(\vec{r},\vec{r}') = \frac{1}{\varepsilon_0}\sum_n \frac{\psi_n(\vec{r})\psi_n^*(\vec{r}\prime)}{\lambda_n}. \tag{8.63}$$

The $G_D(\vec{r},\vec{r}')$ also satisfies the boundary condition, $G_D(\vec{r},\vec{r}')|_{\vec{r}_s} = 0$.

Example 8.1 *Find the Green's function in a cubic box. A spherical cavity was cut in the sphere with a radius a (a < R) as shown in **Figure 8.11**.*

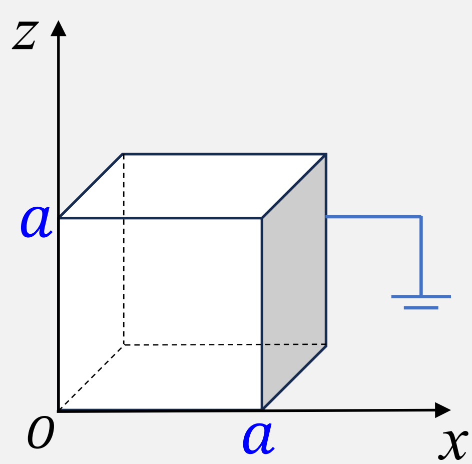

Fig. 8.11 A cube with length of a.

Discussion: We can use the eigenfunction method to determine the Green's function. At the boundary, the potential is set to be zero. According to our boundary value problem knowledge, the eigenfunctions are sine function.

Solution: The eigenfunction for this boundary can be written as,

$$\psi_{lmn}(\vec{r}) = \sin\frac{l\pi x}{a}\sin\frac{m\pi y}{a}\sin\frac{n\pi z}{a}$$

The eigen values are,

$$\lambda_{lmn} = \frac{\pi^2}{a^2}(l^2 + m^2 + n^2)$$

Thus, the Green's function can be written as

$$G_D(\vec{r}, \vec{r}') = \frac{8}{\pi a^2 \varepsilon_0} \sum_{l,m,n=1} \frac{\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{a}\sin\frac{n\pi z}{a}\sin\frac{l\pi x'}{a}\sin\frac{m\pi y'}{a}\sin\frac{n\pi z'}{a}}{l^2 + m^2 + n^2}$$

Example 8.2 *A plane, circular, conducting disk separated with a very thin cut from the remining infinite conductive plane is biased with a potential V_0, while the rest of the plane is grounded. Find the potential in $z \geq 0$ space.*

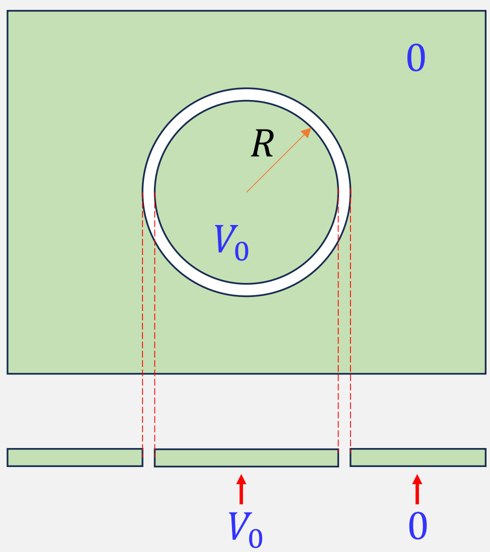

Fig. 8.12 A conductive plate with a circular thin cut.

Discussion: The Green's function can be determined by the method of image. If the gap can be neglected, the problem is an infinitely large conductive plane, so the Green's function can be determined by **Equation 8.7**.

Discussion: Based on **Equation 8.7**, the Green's function can be written as,

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2]^{1/2}} - \frac{1}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z+z')^2]^{1/2}}\right\}.$$

For $z \geq 0$,

$$\left.\frac{\partial G}{\partial n'}\right|_{S'} = \left.\frac{\partial G}{\partial z'}\right|_{z'=0} = \frac{1}{4\pi\varepsilon_0}\frac{2z}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2]^{3/2}}.$$

Due to axial symmetry, we can choose $\phi = 0$. Since there is no charge distribution, we have

$$\varphi(\vec{r}) = \frac{V_0}{4\pi}\oiint_S dS'\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}$$
$$= \frac{V_0 z}{2\pi}\int_0^{2\pi} d\phi' \int_0^R \frac{\rho' d\rho'}{[\rho^2+\rho'^2-2\rho\rho'\cos\phi'+z^2]^{3/2}}.$$

This integration is not solvable. But for $\rho = 0$, one has

$$\varphi(\rho = 0, z) = V_0 z\int_0^R \frac{\rho' d\rho'}{[\rho'^2+z^2]^{3/2}} = V_0[1-\frac{z}{\sqrt{R^2+z^2}}].$$

When $z \to 0$, $\varphi \to V_0$; when $z \to \infty$, $\varphi \approx V_0\frac{R^2}{2z^2}$.

Example 8.3 *A conductive sphere is cut into two halves with an infinitely small gap. The top half has a potential $V_0/2$ and the bottom half has $-V_0/2$. Find the potential outside the sphere.*

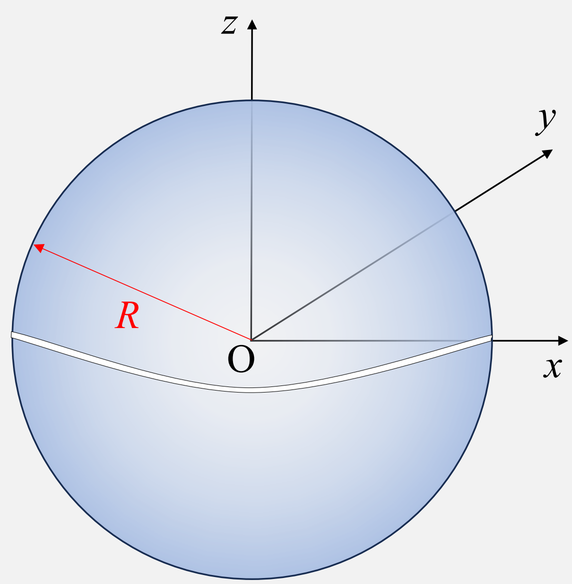

Fig. 8.13 Cut conductive sphere.

Discussion: The Green's function can be determined by the method of image, i.e., via a point charge outside a conductive sphere, which can be determined by **Equation 8.15**.

Discussion: Based on **Equation 8.15**, the Green's function can be written as,

$$G(\vec{r},\vec{r}') = \frac{1}{[r^2+r'^2-2rr'\cos\gamma]^{1/2}} - \frac{R/r'}{[r^2+(R^2/r')^2-2r\,R^2\cos\gamma/r']^{1/2}},$$

with $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. At the boundary $(r = R)$, we have

$$\left.\frac{\partial G}{\partial r'}\right|_{r'=R} = \frac{r^2-R^2}{R[r^2+R^2-2rR\cos\gamma]^{3/2}}.$$

Due to axial symmetry, we can choose $\phi = 0$. Since there is no charge distribution, based on **Equation 8.60**, we have

$$\varphi(\vec{r}) = \frac{1}{4\pi}\oiint_S dS'\frac{\partial G(\vec{r},\vec{r}')}{\partial n'} = \frac{V_0}{2}\left[1-\frac{z^2-R^2}{z\sqrt{R^2+z^2}}\right].$$