Chapter 3
Electrostatics

3.1 Electrostatic Field and Potential

According to the Maxwell's equations, the time invariant $\vec{E}$ and $\vec{D}$ fields should satisfy the following equations,

$$\begin{cases} \nabla \cdot \vec{D} = \rho \\ \nabla \times \vec{E} = 0 \end{cases}. \tag{3.1}$$

So both fields are time invariant. According to Coulomb's law, the electrostatic force $\Delta\vec{F}_E$ generated by an external electric field $\vec{E}$ on a small volume of charge distribution shown in **Figure 3.1a** can be written as (i.e., the charged object is acting as a test object),

$$\Delta\vec{F}_E = \Delta q\vec{E}(\vec{r}) = \rho_2(\vec{r})\vec{E}(\vec{r})\Delta V. \tag{3.2}$$

where $\Delta q = \rho_2(\vec{r})\Delta V$ is the charge of the small volume at location $\vec{r}$, and $\vec{E}(\vec{r})$ is the external field distribution at $\vec{r}$. Thus, the total electrostatic force $\vec{F}_E$ acting on the charge distributed object V_2 is,

$$\vec{F}_E = \iiint_{V_2} \vec{E}(\vec{r})\ \rho_2(\vec{r})dV. \tag{3.3}$$

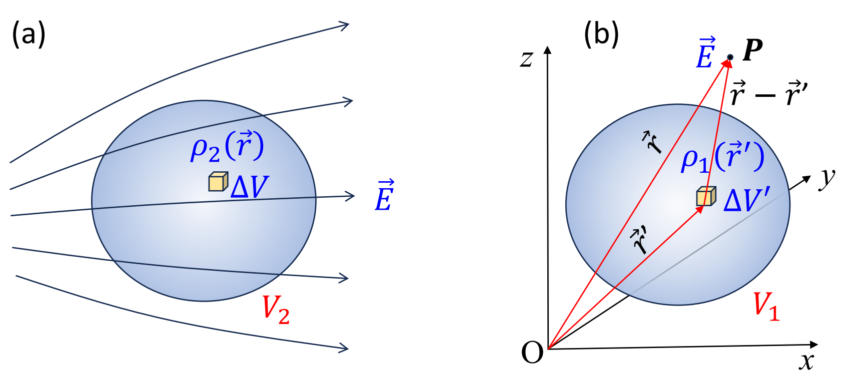

Fig. 3.1 (a) A charged object V_2 in an electric field $\vec{E}$ and (b) the electric field $\vec{E}$ generated by a charged object V_1.

And the torque $\vec{N}$ acting on the object V_2 can be written as

$$\Delta\vec{N} = \vec{r} \times \Delta\vec{F}_E,$$

and,

$$\vec{N} = \iiint_{V_2} \vec{r} \times \vec{E}(\vec{r})\ \rho_2(\vec{r})dV. \tag{3.4}$$

For two-point charges, according to Coulomb's law,

$$\vec{F}_E(\vec{r},\vec{r}') = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}. \tag{3.5}$$

Where $\vec{r}'$ is the location of the source charge q_1 and $\vec{r}$ is the location of the test charge q_2. The electric field $\vec{E}_1$ produced by the source charge q_1 at location $\vec{r}$ can be written as,

$$\vec{E}_1 = \frac{q_1}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}. \tag{3.6}$$

For a bulk charge distribution V_1 as shown in **Figure 3.1b**, the electric field $\vec{E}$ generated at location $\vec{r}$ can be expressed as,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_{V_1} \rho_1(\vec{r}')\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}dV'. \tag{3.7}$$

Similarly, if an object only has a surface charge distribution $\sigma(\vec{r}_s)$, where $\vec{r}_s$ indicate all locations of the surface, the electric field $\vec{E}$ generated at location $\vec{r}$ is,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iint_S \sigma(\vec{r}_s)\frac{\vec{r}-\vec{r}_s}{|\vec{r}-\vec{r}_s|^3}dS', \tag{3.8}$$

while a linear charge distribution with a linear charge density $\lambda(\vec{r}_l)$, the field can be expressed as,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \lambda(\vec{r}_l)\frac{\vec{r}-\vec{r}_l}{|\vec{r}-\vec{r}_l|^3}dl'. \tag{3.9}$$

Therefore, based on **Equations 3.3** and **3.7**, for two charged objects V_1 and V_2, since the object V_1 can generate a field $\vec{E}$ at the location of the object V_2, the electrostatic force $\vec{F}_{1\to2}$ acting on V_2 can be written as,

$$\vec{F}_{1\to2} = \frac{1}{4\pi\varepsilon_0}\iiint_{V_2} \rho_2(\vec{r})dV \iiint_{V_1} \rho_1(\vec{r}')\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\ dV'. \tag{3.10}$$

Since $\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\nabla\frac{1}{|\vec{r}-\vec{r}'|}$, $\nabla\times\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_V \rho(\vec{r}')\nabla\times\left[\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right]dV' = 0$, mathematically $\vec{E}(\vec{r})$ can be expressed as a gradient of a scalar function. Physically, $\vec{E}(\vec{r})$ is directly related to the electrostatic potential $\varphi(\vec{r})$ generated by a source object (or source electric field),

$$\vec{E} = -\nabla\varphi(\vec{r}). \tag{3.11}$$

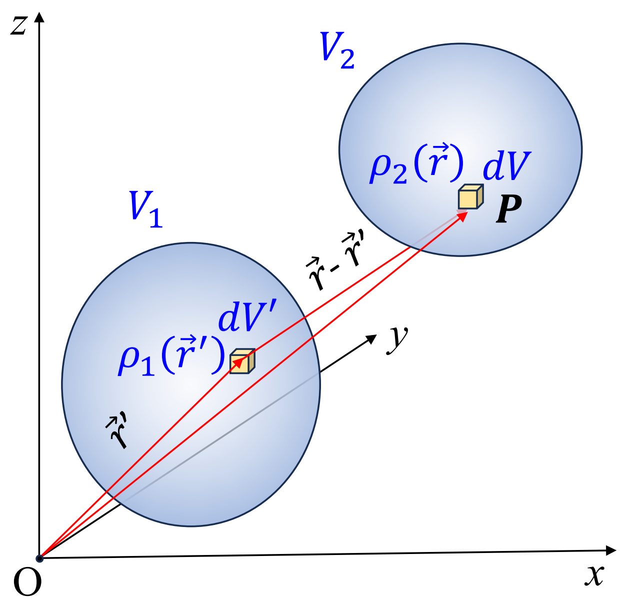

Fig. 3.2 The interaction of two charged objects V_1 and V_2.

Note that the electric potential $\varphi(\vec{r})$ does not have an absolute value, rather a reference point $\vec{r}_0$ must be selected, so that $\varphi(\vec{r}_0) = 0$, and all potential at any arbitrary location is a relative value with respect to $\varphi(\vec{r}_0)$. General convention is that $\varphi(\vec{r}) = 0$ when $\vec{r} \to \infty$. For a point charge q_1, the potential is written as,

$$\varphi(\vec{r}) = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}\prime|}. \tag{3.12}$$

For a bulk charge distribution V_1, the potential $\varphi(\vec{r})$ can be expressed as,

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_{V_1}\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}\prime|}dV', \tag{3.13}$$

Similar expressions can be obtained for both surface and linear charge distributions as

$$\begin{cases} \varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\sigma(\vec{r}_s)}{|\vec{r}-\vec{r}_s|}dS' & \text{for surface charge} \\ \varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda(\vec{r}_l)}{|\vec{r}-\vec{r}_l|}dl' & \text{for linear charge} \end{cases}. \tag{3.14}$$

Combining the definition of $\varphi(\vec{r})$ and the Gauss law, we have,

$$\nabla^2\varphi(\vec{r}) = -\rho/\varepsilon_0. \tag{3.15}$$

This is the Poisson's equation for electrostatics. When $\rho = 0$, the Poisson's equation reduces to the Laplace equation,

$$\nabla^2 \varphi(\vec{r}) = 0. \tag{3.16}$$

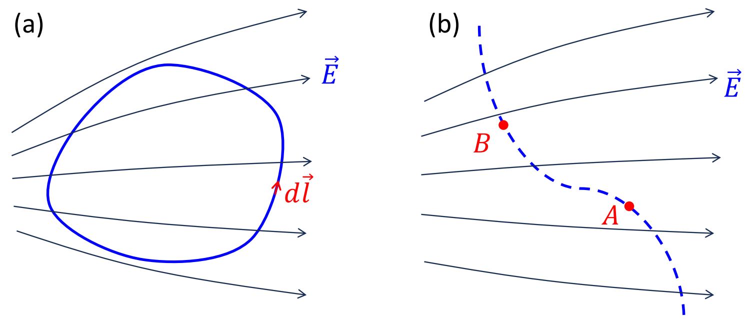

Fig. 3.3 (a) A loop integration of $\vec{E}$ and (b) a path integration of a charged particle moving from Location A to Location B.

Both **Equations 3.16** and **3.15** are the two most important equations for electrostatics. In **Chapters 7** and **8**, we will give detailed discussions on how to solve the boundary-value problems for these two equations.

One important property for electrostatic force is that it is conservative since $\vec{F}_E = q\vec{E}$ and $\nabla \times \vec{E} = 0$, i.e., the static electric field lines do not cross each other. For any loop integration of $\vec{E}$ as shown in **Figure 3.3a**, according to Stokes theorem,

$$\oint_L \vec{E} \cdot d\vec{l} = \iint_S \left(\nabla \times \vec{E}\right) \cdot \hat{n} dS' = 0, \tag{3.17}$$

where $\hat{n}$ is the surface normal of the area in the loop in **Figure 3.3a**. Thus, the total work ΔW done by the electrostatic force $\vec{F}_E$ along the loop in **Figure 3.3a** can be written as,

$$\Delta W = \oint_L \vec{F}_E \cdot d\vec{l} = q \oint_L \vec{E} \cdot d\vec{l} = 0. \tag{3.18}$$

i.e., $\Delta W = 0$, which is a condition to show that the force $\vec{F}_E$ is conservative. Alternatively as shown in **Figure 3.3b**, the work done ΔW by an electrostatic force $\vec{F}_E$ to move a charged particle from Location A to Location B is,

$$\begin{aligned} \Delta W &= \int_A^B \vec{F}_E \cdot d\vec{l} = \int_A^B q\vec{E} \cdot d\vec{l} \\ &= -\int_A^B q\nabla\varphi \cdot d\vec{l} = -q(\varphi_B - \varphi_A). \end{aligned} \tag{3.19}$$

Thus, the work ΔW only depends on the electric potentials at Location A to Location B, not on a specific the path how the particle is moved. Such a fact also demonstrates that the force $\vec{F}_E$ is conservative. Therefore, both **Equations 3.18** and **3.19** illustrate that the electrostatic force is a conservative force.

3.2 Gauss Law and Boundary Conditions

For a static electric field $\vec{E}$, the Gauss law can be written as,

$$\oiint_S \vec{E}\cdot\hat{n}dS' = \frac{q_{in}}{\varepsilon_0}, \tag{3.20}$$

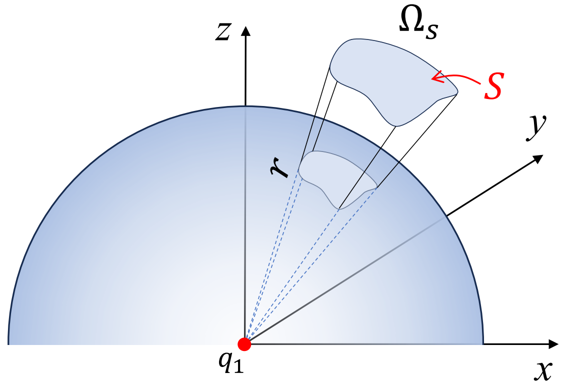

Fig. 3.4 An illustration of the solid angle Ω_s.

where q_{in} is the total net charges enclosed in the closed surface. It can be easily used to find the electric field of symmetrically distributed charged object. In fact, the integration in the right-hand side of **Equation 3.20** is the total electric flux passing through the enclosed surface. However, for any arbitrary shaped curved surface S as shown in **Figure 3.4**, the electric flux generated by a point charge q_1 in the origin of the coordinate can be written as

$$\Phi_E = \iint_S \vec{E}\cdot\hat{n}dS' = \frac{q_1}{4\pi\varepsilon_0}\iint_S \frac{\hat{r}}{r^2}\cdot\hat{n}dS' = \frac{q_1}{4\pi\varepsilon_0}\Omega_s, \tag{3.21}$$

where Ω_s is the solid angle covered by the area S as shown in **Figure 3.4**, i.e.,

$$\Omega_s = \iint_S \frac{dS\prime}{r^2}. \tag{3.22}$$

Since $dS' = \hat{r}\cdot\hat{n}dS'$, in the Spherical Coordinates, one has

$$d\Omega_s = \frac{\hat{r}\cdot\hat{n}dS\prime}{r^2} = \sin\theta\, d\theta d\phi. \tag{3.23}$$

Based on Gauss's law and **Equation 3.1**, if there is a material boundary to split the space into two parts as shown in **Figure 3.5**, we can construct a tiny cylindrical Gauss surface across the boundary and apply **Equation 3.20** to investigate the relationship between $\vec{E}_1$ and $\vec{E}_2$ near the boundary. Similarly, a tiny rectangular loop can be established across the boundary and apply **Equation 3.17** for the fields $\vec{E}_1$ and $\vec{E}_2$ near the boundary, we have

$$\begin{cases} \left(\vec{E}_2 - \vec{E}_1\right)\cdot\hat{n}_2 = \frac{\sigma_s}{\varepsilon_0} \\ \left(\vec{E}_2 - \vec{E}_1\right)\times\hat{n}_2 = 0 \end{cases}, \tag{3.24}$$

where σ_s is the surface charge density on the boundary. **Equation 3.24** shows that the tangential component of the electric field is continuous at the boundary while the normal component may not, depending on whether there is any net charge distribution on the boundary. These boundary matching conditions can be translated to electrostatic potential $\varphi(\vec{r})$ near the boundary, with

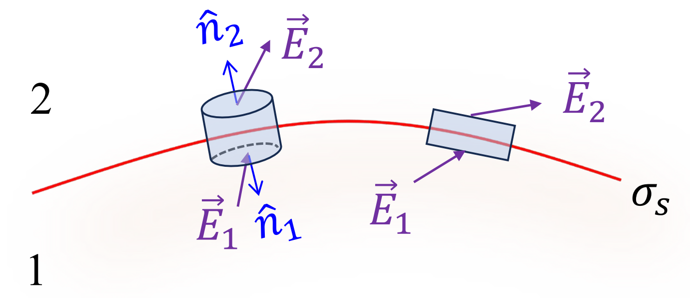

Fig. 3.5 An illustration of boundary, associated Gauss surface, loop across the boundary and electric fields.

$$\begin{cases} \frac{\partial \varphi_2}{\partial n} - \frac{\partial \varphi_1}{\partial n} = \frac{\sigma}{\varepsilon_0} \\ \varphi_2(\vec{r}_s) = \varphi_1(\vec{r}_s) \end{cases}, \tag{3.25}$$

where $\vec{r}_s$ is the interface location. The bottom expression in **Equation 3.25** demonstrates that the electrostatic potential $\varphi(\vec{r})$ at the boundary shall be continuous.

Example 3.1 *A uniformly charged sphere with a charge density ρ and a radius of R centered at O. A spherical cavity was cut in the sphere with a radius a (a < R) as shown in **Figure 3.6**. (1) Find the electric field at P(x, y, z) location; (2) Find the electric static potential at P.*

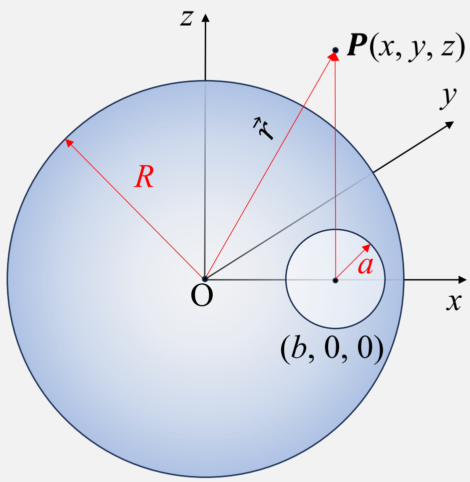

Fig. 3.6 A uniformly charged sphere with a spherical cavity.

Discussion: This problem can be solved using the superposition of the electric field and potential. The cavity system can be treated as two charged objects, one is a uniformly charged sphere S_1 with a radius of R, located at $(0,0,0)$, and charge density of ρ; the other is a uniformly charged sphere S_2 with a radius of a, located at $(b, 0,0)$, and charge density of $-\rho$. The sphere S_1 will generate an electric field $\vec{E}_1$ at location P outside S_1, similarly, the sphere S_2 will generate an electric field $\vec{E}_2$ at location P, so the total electric field $\vec{E}_P$ at P is,

$$\vec{E}_P = \vec{E}_1 + \vec{E}_2.$$

Since the sphere S_1 has a spherical symmetry, we can construct a Gauss sphere centered at $(0,0,0)$ and passing P, and apply the Gauss's law (**Equation 3.20**) to find $\vec{E}_1$,

$$\vec{E}_1 = \frac{R^3\rho}{3\varepsilon_0}\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}}.$$

Similarly for the sphere S_2, we can construct a Gauss sphere centered at $(b, 0,0)$ and passing P to obtain $\vec{E}_2$,

$$\vec{E}_2 = -\frac{a^3\rho}{3\varepsilon_0}\frac{(x-b)\hat{x}+y\hat{y}+z\hat{z}}{[(x-b)^2+y^2+z^2]^{3/2}}.$$

Therefore,

$$\vec{E}_P = \frac{R^3\rho}{3\varepsilon_0}\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}} - \frac{a^3\rho}{3\varepsilon_0}\frac{(x-b)\hat{x}+y\hat{y}+z\hat{z}}{[(x-b)^2+y^2+z^2]^{3/2}}.$$

The electric potential φ_1 generated by S_1 at P can be written as,

$$\varphi_1 = \frac{R^3\rho}{3\varepsilon_0}\frac{1}{(x^2+y^2+z^2)^{1/2}},$$

And φ_2 generated by S_2 at P is,

$$\varphi_2 = -\frac{a^3\rho}{3\varepsilon_0}\frac{1}{[(x-b)^2+y^2+z^2]^{1/2}}.$$

Therefore the total potential φ_P at P is

$$\varphi_P = \varphi_1 + \varphi_2 = \frac{R^3\rho}{3\varepsilon_0}\frac{1}{(x^2+y^2+z^2)^{1/2}} - \frac{a^3\rho}{3\varepsilon_0}\frac{1}{[(x-b)^2+y^2+z^2]^{1/2}}.$$

Example 3.2 *The charge q is uniformly distributed on a spherical conductor with a radius R. (1) Prove that the force felt by a small charged surface dq on the conductor follows* $d\vec{F} = \frac{1}{2}Edq\hat{n}$*, here* $E = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}$*, here* $\hat{n}$ *is the surface normal of the dq area. (2) If the sphere is cut a half, and one still wants to keep these two halves together, how much external force one shall apply?*

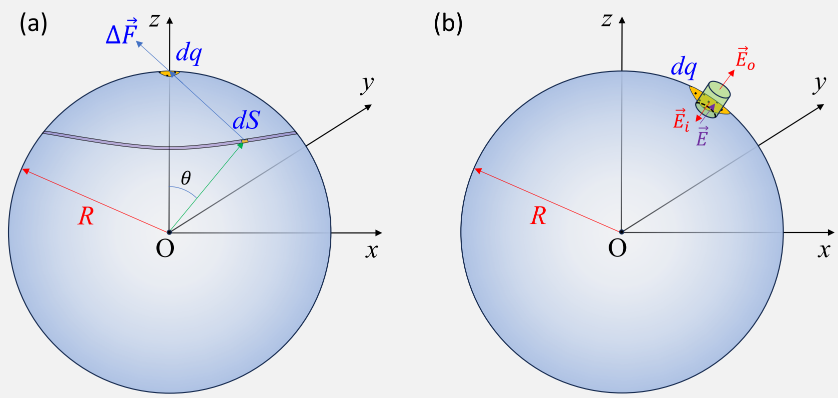

Fig. 3.7 A spherical conductor and a small surface area.

Discussion: For a metallic sphere, the charge only distributed on the surface of the sphere, and the electric field inside the sphere and near the *dq* surface is zero. This zero field can be viewed as the superposition of the surface charge generated electric field $\vec{E}_i$ pointing into the center of the sphere, and the electric field $\vec{E}_e$ generated by the rest of the charged area of the sphere except for *dq* area as shown in **Figure 3.7b**, i.e.,

$$\vec{E}_i + \vec{E}_e = 0$$

Around *dq* area, construct a tiny cylindrical Gauss surface as shown in **Figure 3.7b**, we have

$$E_i dS + E_o dS = \frac{\sigma dS}{\varepsilon_0},$$

where $\sigma = \frac{q}{4\pi R^2}$. Near the surface, $E_i = E_o$, thus,

$$E_i = \frac{\sigma}{2\varepsilon_0}.$$

That is, $E_e = E_i = \frac{\sigma}{2\varepsilon_0}$, and the field is pointing outward from the sphere surface. Thus, the force acting onto this small dq area due to the rest of the charge distribution is,

$$d\vec{F} = \frac{\sigma}{2\varepsilon_0} dq\hat{r} = \frac{q}{8\pi R^2} dq\hat{r} = \frac{1}{2} E(R) dq\hat{r},$$

where $E(R) = \frac{q}{4\pi R^2}$.

For the second part of the problem, since we already know the solution of first part of the problem, go back to **Figure 3.7a**, consider the force acting on the top hemisphere, only the z-component force will not be canceled due to the azimuthal symmetry, since $dq = \sigma dS$ thus,

$$F = \iint dF \sin\theta \sin\phi = \frac{\sigma^2}{2\varepsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} R^2 \sin^2\theta \sin\phi \, d\theta.$$

Therefore,

$$F = \frac{\pi\sigma^2 R^2}{2\varepsilon_0} = \frac{q^2}{32\pi\varepsilon_0 R^2}.$$

This is the force to keep the two hemispheres together.

3.3 Electrostatic Potential Energy

According to **Figure 3.2**, a charge distribution V_1 will generate a potential $\varphi_1(\vec{r})$ at a charge distribution V_2, and the electrostatic interaction potential energy U_I between the two objects can be expressed as,

$$U_I = q\varphi_E = \iiint_{V_2} \rho_2(\vec{r})\varphi_1(\vec{r}) dV. \tag{3.26}$$

Since $\varphi_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_{V_1} \frac{\rho_1(\vec{r}')}{|\vec{r}-\vec{r}'|} dV'$, thus,

$$U_I = \frac{1}{4\pi\varepsilon_0} \iiint_{V_2} \rho_2(\vec{r}) dV \iiint_{V_1} \frac{\rho_1(\vec{r}')}{|\vec{r}-\vec{r}'|} dV'$$

$$= \frac{1}{4\pi\varepsilon_0} \iiint_{V_2} dV \iiint_{V_1} \frac{\rho_1(\vec{r}')\rho_2(\vec{r})}{|\vec{r}-\vec{r}'|} dV'$$

$$= \frac{1}{4\pi\varepsilon_0} \iiint_{V_1} \rho_1(\vec{r}') dV' \iiint_{V_2} \frac{\rho_2(\vec{r})}{|\vec{r}-\vec{r}'|} dV = \iiint_{V_1} \rho_1(\vec{r}')\varphi_2(\vec{r}') dV',$$

that is,

$$\iiint_{V_2} \rho_2(\vec{r})\varphi_1(\vec{r}) dV = \iiint_{V_1} \rho_1(\vec{r}')\varphi_2(\vec{r}') dV'. \tag{3.27}$$

Equation 3.27 is called the Green's reciprocity relationship, i.e., the potential energy of $\rho_2(\vec{r})$ in the field produced by $\rho_1(\vec{r}')$ equals to the potential energy of $\rho_1(\vec{r}')$ in the field produced by $\rho_2(\vec{r})$.

However, the electrostatic total potential energy U_T of the two-object system not only include the interaction energy U_E, but also shall include the electrostatic self potential energies U_{S1} and U_{S2} of V_1 and V_2, because in order to build a charged object, we have to overcome the electrostatic force to do work to bring a unit charge into an object. Let's consider how to calculate the electrostatic self-energy U_S of an *N*-charged particle system.

(1) Bring one particle q_1 to the space, the work W_1 done by the electrostatic force is 0.

(2) Bring the 2nd charge q_2 into the space, the work W_2 done by the electrostatic force is,

$$W_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}_1-\vec{r}_2|}. \tag{3.28}$$

(3) Bring the 3rd charge q_3 into the space, the work W_3 done by the electrostatic force is (since there are two charges already in space)

$$W_3 = \frac{q_3}{4\pi\varepsilon_0}\left[\frac{q_1}{|\vec{r}_3-\vec{r}_1|}+\frac{q_2}{|\vec{r}_3-\vec{r}_2|}\right]. \tag{3.29}$$

So the self-energy U_S of the 3-particle system is written as,

$$U_S = W_1 + W_2 + W_3 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}_1-\vec{r}_2|}+\frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{|\vec{r}_2-\vec{r}_3|}+\frac{1}{4\pi\varepsilon_0}\frac{q_3 q_1}{|\vec{r}_3-\vec{r}_1|}. \tag{3.30}$$

(4) Bring the 4th charge q_4 into the space, U_S can be written as

$$U_S = \sum_{j=1}^{4} W_j = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{4}\left(\sum_{k<j}^{4}\frac{q_j q_k}{|\vec{r}_j-\vec{r}_k|}\right). \tag{3.31}$$

Therefore, the general expression for U_S of an N-particle system can be written as,

$$U_S = \sum_{j=1}^{N} W_j = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{N}\left(\sum_{k<j}^{N}\frac{q_j q_k}{|\vec{r}_j-\vec{r}_k|}\right), \tag{3.32}$$

Or

$$U_S = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{N}\left(\sum_{k\neq j}^{N}\frac{q_j q_k}{|\vec{r}_j-\vec{r}_k|}\right). \tag{3.33}$$

The factor $\frac{1}{2}$ is counting for the repeated permutations.

According to **Equation 3.33**, for a continuum charge distribution shown in **Figure 3.8**, the electrostatic self-energy U_S can be written as,

$$U_S = \frac{1}{2}\iiint_V \rho(\vec{r})\varphi(\vec{r})dV, \tag{3.34}$$

where $\varphi(\vec{r})$ is the electrostatic potential generated by the rest of the charge distribution of the object. Since $\nabla\cdot\vec{D} = \rho(\vec{r})$, i.e., $\rho(\vec{r}) = \varepsilon_0\nabla\cdot\vec{E} = -\varepsilon_0\nabla^2\varphi(\vec{r})$ inserting this expression to **Equation 3.34**,

$$U_S = -\frac{\varepsilon_0}{2}\iiint_V [\nabla^2\varphi(\vec{r})]\varphi(\vec{r})dV. \tag{3.35}$$

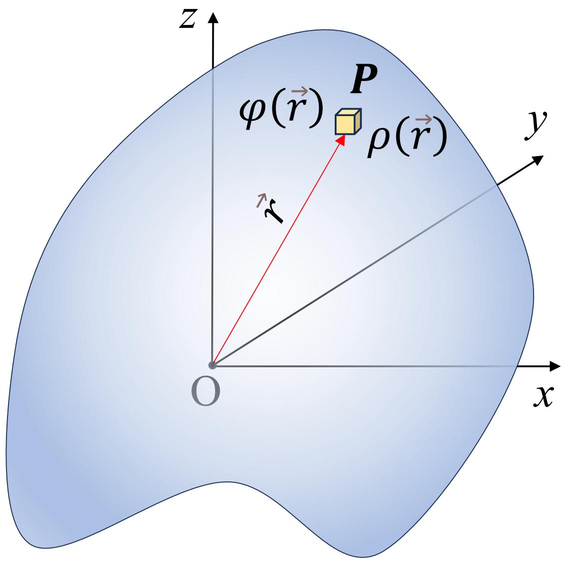

Fig. 3.8 The configuration to calculate the self-energy U_S of a continuum charge distribution.

Integrating **Equation 3.35** by parts,

$$U_S = \frac{\varepsilon_0}{2}\iiint_V |\nabla\varphi(\vec{r})|^2 dV = \frac{\varepsilon_0}{2}\iiint_V \left|\vec{E}(\vec{r})\right|^2 dV. \quad (3.36)$$

Based on **Equation 3.36**, we can define an electrostatic energy density u_E,

$$u_E = \frac{\varepsilon_0}{2}\left|\vec{E}(\vec{r})\right|^2. \quad (3.37)$$

The integration in **Equation 3.36** should go through the entire space of where $\vec{E}(\vec{r})$ is occupied, which means that it can cover to infinite if the $\vec{E}(\vec{r})$ generated by the object can extend to infinite. Based on **Equations 3.26** and **3.36,** the total electrostatic energy U_T of two charged distributed objects shown in **Figure 3.2** can be written into three parts,

$$U_T = U_{S1} + U_{S2} + U_I, \quad (3.38)$$

where

$$U_I = \frac{1}{4\pi\varepsilon_0}\iiint_{V_2} dV \iiint_{V_1} \frac{\rho_1(\vec{r}')\rho_2(\vec{r})}{|\vec{r}-\vec{r}'|} dV', \quad (3.26)$$

$$U_{S1} = \frac{1}{2}\iiint_{V_1} \rho_1(\vec{r}')\varphi_1(\vec{r}')dV, \quad (3.29)$$

$$U_{S2} = \frac{1}{2}\iiint_{V_2} \rho_2(\vec{r})\varphi_2(\vec{r})dV. \quad (3.40)$$

Both $\varphi_1(\vec{r}')$ and $\varphi_2(\vec{r})$ are electrostatic potentials generated by V_1 and V_2, respectively.

Example 3.3 *Find the self-energy of a uniformly charge distributed sphere with a charge density ρ and a radius R.*

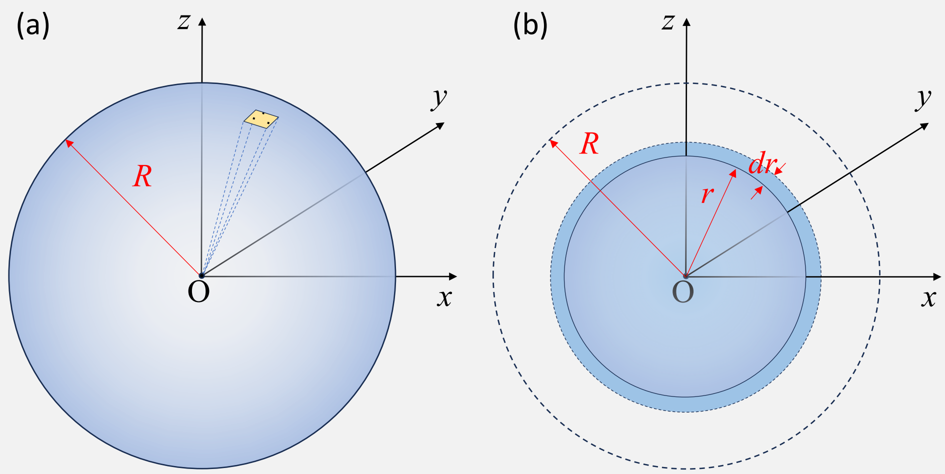

Fig. 3.9 (a) A sphere of uniformly distributed charges. (b) Building the charged sphere layer-by-layer (each layer is a charged shell).

Discussion: The uniformly charged sphere is shown in **Figure 3.9a**, with a total charge of $q = \frac{4}{3}\pi R^3 \rho$. There are three different ways to solve this problem.

<u>*Mathod#1*</u>: Based on **Equation 3.36**, we can first calculate the electric field generated by the charged sphere. Based on the Gauss's law, the E-field $E(r)$ can be calculated as,

$$E(r) = \begin{cases} \frac{q}{4\pi\varepsilon_0}\frac{r}{R^3} & \text{for } r \le R \\ \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2} & \text{for } r > R \end{cases}.$$

Therefore,

$$U_S = \frac{\varepsilon_0}{2}\int_0^\infty E(r)^2\, 4\pi r^2 dr$$

$$= \frac{\varepsilon_0}{2}\int_0^R \left(\frac{q}{4\pi\varepsilon_0}\frac{r}{R^3}\right)^2 4\pi r^2 dr + \frac{\varepsilon_0}{2}\int_R^\infty \left(\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\right)^2 4\pi r^2 dr$$

$$= \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2\left(\frac{4\pi}{5R} + \frac{4\pi}{R}\right) = \frac{3}{5}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{R}.$$

<u>*Mathod#2*</u>: Based on **Equation 3.34**, we can first calculate the electric potential generated by the charged sphere,

$$\varphi(r) = \begin{cases} \frac{q}{4\pi\varepsilon_0}\left(\frac{3}{2R} - \frac{r^2}{2R^3}\right) & \text{for } r \le R \\ \frac{q}{4\pi\varepsilon_0}\frac{1}{r} & \text{for } r > R \end{cases}.$$

Since $\rho(r) = 0$ for $r > R$, we only need to calculate the integration for $r \le R$,

$$U_S = \frac{1}{2}\int_0^R \frac{q}{4\pi\varepsilon_0}\rho\left(\frac{3}{2R} - \frac{r^2}{2R^3}\right) 4\pi r^2 dr = \frac{3q^2}{16\pi\varepsilon_0 R^3}\left(R^2 - \frac{R^2}{5}\right) = \frac{3}{5}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{R}.$$

Mathod#3: We can build the charged sphere layer by layer as shown in **Figure 3.9b**. Assuming that a small charged sphere with radius r, the total charge of this sphere is $q = \frac{4}{3}\pi r^3\rho$, and the field and potential outside the small charged sphere can be viewed as those generated by a point charge *q* in the center. In order to build a charged sphere with a radius *R*, a new layer of charged shell with radius *r* and thickness *dr*, with a total charge of $dq = 4\pi r^2\rho dr$, needs to be brought from infinity to locate *r*, where an external force needs to do a positive work to overcome the repulsion force between the small sphere and the shell. We will keep on adding the shells till the radius *r* reaches *R*. Therefore, the total work done by the external force will be the self-energy of the uniformly charged sphere, i.e.,

$$U_S = \int_0^R \varphi(r)dq = \int_0^R \frac{q}{4\pi\varepsilon_0}\frac{1}{r} 4\pi r^2 \rho dr = \frac{4\pi\rho^2}{3\varepsilon_0}\int_0^R r^4 dr = \frac{3}{5}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{R}.$$

In-class Activity

3-1. A conducting object has a hollow cavity in its interior. If a point charge *q* is introduced into the cavity, prove that the charge -*q* is introduced on the surface of the cavity.

3-2. Please verify the E-filed boundary matching conditions for a charged conductor sphere.

3-3. Given a spherical shell of charge, radius *R*, uniform surface charge density σ_0. Determine the self-energy of the charge distribution.

3-4. Given two charged spheres, one with charge q_1 and radius R_1, and the other with charge q_2 and radius R_2. The two spheres are placed at a center-to-center distance of *l*. Find the total electrostatic energy of the system.