Chapter 5
Conductors

5.1 General Electrostatic Property of a Conductor

5.1.1 Charge distribution on a conductor

Thomson's Theorem of Electrostatics states that the electrostatic energy of a conductor of fixed shape and size is minimized when its charge q distributes itself in such a way that the electrostatic potential φ remains constant throughout the entire object. This uniform potential implies that the electric field $\vec{E}$ inside the conductor is zero, as linked by the relationship $\vec{E} = -\nabla\varphi = 0$. This fundamental property is crucial because it means that there is no electric field acting on the charges within the conductor, ensuring that they are in electrostatic equilibrium. Consequently, any excess charges in a conductor should be distributed on the surface of the conductor. If any excess charge were to reside inside the conductor, it would create an electric field that would cause the charges to move until they reach the surface.

When we revisit the boundary conditions governing the electric field at the surface of the conductor, we refer to **Equation 3.24**. Since the electric field $\vec{E}_{in}$ inside the conductor is zero, then the electric field $\vec{E}_{out}$ outside the conductor, particularly near its surface, may not be zero. The relevant boundary conditions can be summarized as,

$$\begin{cases} \vec{E}_{out} \cdot \hat{n}_2 = \frac{\sigma_s}{\varepsilon_0} \\ \vec{E}_{out} \times \hat{n}_2 = 0 \end{cases}. \tag{5.1}$$

In this expression, σ_s represents the surface charge density of the conductor, and $\hat{n}_2$ is the outward normal vector at the surface. The first expression in **Equation 5.1** indicates that the normal component of $\vec{E}_{out}$ just outside the conductor surface is not zero and its magnitude is proportional to the surface charge density σ_s, while the second expression signifies that the electric field $\vec{E}_{out}$ has no tangential component at the surface. Therefore, $\vec{E}_{out}$ can be expressed as,

$$\vec{E}_{out} = \frac{\sigma_s}{\varepsilon_0}\hat{n}_2 \,. \tag{5.2}$$

This result demonstrates that the electric field outside a conductor always points outward, normal to the surface, and its magnitude is proportional to the surface charge density, as illustrated in **Figure 5.1**.

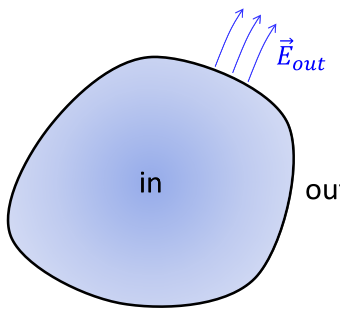


Fig. 5.1 Boundary of a conductor.

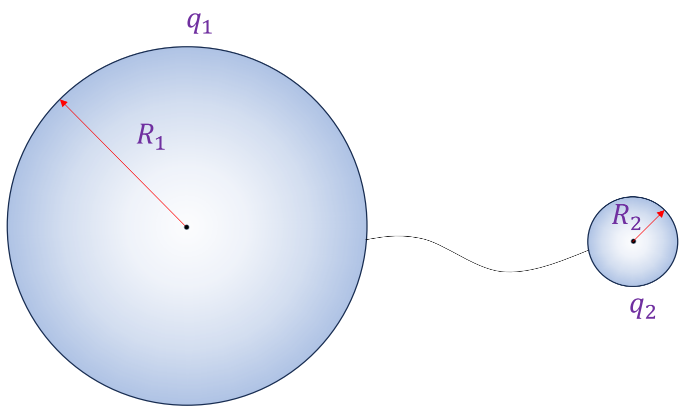


Fig. 5.2 Two conducting spheres connected by a thin conducting wire.

Next, we can analyze the surface charge density and the electric field for a charged conductor. When a total charge q is placed on a conductor with an arbitrary shape, the distribution of charge on its surface is influenced by the local curvature κ of the surface, where $\kappa = \frac{1}{R}$. A higher curvature results in a greater surface charge density σ.

To demonstrate this concept, let's consider two conducting spheres with radii R_1 and R_2, which are connected by a thin conducting wire as shown in **Figure 5.2**. When a total charge q is placed on the system, it will redistribute between two spheres. If we assume that there is no charge retained in the wire, we can assign charges q_1 and q_2 to spheres R_1 and R_2, respectively. According to the conservation of charge, the total charge satisfies the equation,

$$q_1 + q_2 = q. \tag{5.3}$$

The electric potentials φ_1 and φ_2 on the surfaces of spheres R_1 and R_2 are given by,

$$\begin{cases}\varphi_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{R_1} \\ \varphi_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{R_2}\end{cases}. \tag{5.4}$$

Since the two conducting spheres are connected by a thin metal wire, they must be at the same electric potential, i.e., $\varphi_1 = \varphi_2$. Thus, we have the relationship,

$$\frac{q_1}{R_1} = \frac{q_2}{R_2}. \tag{5.5}$$

By combining **Equations 5.3** and **5.4**, we can solve for q_1 and q_2

$$\begin{cases}q_1 = \frac{q}{R_1+R_2}R_1 \\ q_2 = \frac{q}{R_1+R_2}R_2\end{cases}. \tag{5.6}$$

Therefore, the electric field E_1 and E_2 near the surface of each sphere can be written as,

$$\begin{cases}E_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{R_1^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R_1+R_2}\frac{1}{R_1} \\ E_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{R_2^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R_1+R_2}\frac{1}{R_2}\end{cases}. \tag{5.7}$$

This leads us to conclude that,

$$E_1 \propto \frac{1}{R_1} = \kappa_1 \text{ and } E_2 \propto \frac{1}{R_2} = \kappa_2. \tag{5.8}$$

Thus, we see that the electric field intensity near the surface of a conductor is significantly affected by the curvature of its surface. According to **Equation 5.2**, we know that $E_1 \propto \sigma_1$ and $E_2 \propto \sigma_2$. Based on our previous findings, we can relate the surface charge densities to curvature,

$$\sigma_1 \propto \kappa_1 \text{ and } \sigma_2 \propto \kappa_2. \tag{5.9}$$

This relationship demonstrates that conductors with sharper curvatures will exhibit higher surface charge densities and, consequently, stronger electric fields.

5.1.2 The electrostatic induction

Electrostatic induction is a fundamental concept in understanding how conductors respond to external electric influences and plays a crucial role in various electrical phenomena and device operations. When a conductor is placed in an external electric field $\vec{E}_{ext}$, the free charges within the conductor experience a redistribution and displacement. This process leads to a concentration of charges on the surface of the conductor, as illustrated in **Figure 5.3**. This phenomenon, known as electrostatic induction, is pivotal for understanding how conductors shield their interiors from external electric influences.

As a result of the charge redistribution, the electric field inside the conductor becomes zero. The surface charges create an opposing or induced electric field $\vec{E}_{ind}$ that cancels out the external field within the conductor, thus establishing a state of electrostatic equilibrium. Based on **Figure 5.3**, the total electric field $\vec{E}(\vec{r}')$ at a point $\vec{r}'$ inside the conductor can be expressed as,

$$\vec{E}(\vec{r}') = \vec{E}_{ext}(\vec{r}') + \vec{E}_{ind}(\vec{r}')$$
$$= \vec{E}_{ext}(\vec{r}') + \frac{1}{4\pi\varepsilon_0}\iint_S \sigma(\vec{r}_s)\frac{\vec{r}'-\vec{r}_s}{|\vec{r}'-\vec{r}_s|^3}dS' = 0, \quad (5.10)$$

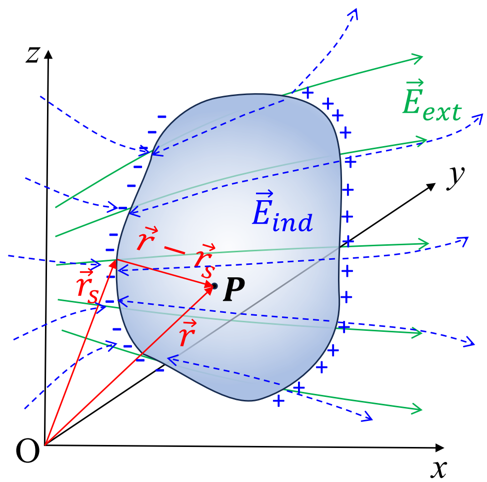


Fig. 5.3 An induced charge distribution on the surface of a conductor and the corresponding induced electric field (blue dashed field lines).

where $\vec{r}_s$ indicates the locations of the surface charges on the conductor. This equation highlights how the induced electric field arises from the distribution of surface charge density σ. The induced field $\vec{E}_{ind}(\vec{r})$ ***outside*** the conductor but produced by the induced surface changes can be expressed as,

$$\vec{E}_{ind}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iint_S \sigma(\vec{r}_s)\frac{\vec{r}-\vec{r}_s}{|\vec{r}-\vec{r}_s|^3}dS'. \quad (5.11)$$

The corresponding induced electric potential $\varphi_{ind}(\vec{r})$ can be written as,

$$\varphi_{ind}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iint_S \sigma(\vec{r}_s)\frac{1}{|\vec{r}-\vec{r}_s|}dS'. \quad (5.12)$$

For points far away from the conductor, where $|\vec{r}-\vec{r}_s| \gg R$ with R being the maximum radius of the conductor), **Equation 5.12** can be approximated in multipole form,

$$\varphi_{ind}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}_s|} + \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot(\vec{r}-\vec{r}_s)}{|\vec{r}-\vec{r}_s|^3} + \cdots \quad (5.13)$$

Here q is the total charge on the conductor's surface. If the conductor is initially neutral, the total induced charge $q = 0$, leading to,

$$\varphi_{ind}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot(\vec{r}-\vec{r}_s)}{|\vec{r}-\vec{r}_s|^3} + \cdots \quad (5.14)$$

Thus, the induced potential consists of contributions from dipoles, quadrupoles, and higher-order multipoles that are induced on the conductor by the external electric field. The primary contribution typically arises from the induced dipole. According to **Equation 4.28**, the induced dipole moment $\vec{p}$ can be defined as,

$$\vec{p} = \iint_S \sigma(\vec{r}_s)\vec{r}_s \, dS'. \tag{5.15}$$

The induced potential $\varphi_{ind}(\vec{r})$ generates an electric field on the surface of the conductor, influencing its interactions with surrounding charges and fields.

In conclusion, electrostatic induction highlights the interplay between conductors and external electric fields. According to previous discussions in **Sections 5.1.1** and **5.1.2**, the surface charge density on a conductor is influenced by two primary factors: the shape of the conductor and the strength of the applied external electric field. The shape determines how charges distribute, while the external field dictates the magnitude and direction of the induced surface charge density. Understanding these relationships is essential for designing and optimizing electrical devices that rely on electrostatic principles.

Example 5.1 *Find the electric potential produced by a conducting sphere of radius R in a uniform external electric field $\vec{E}_0$. (1) Show that the sphere acquires a dipole moment, $\vec{p} = \alpha\varepsilon_0\vec{E}_0$ where $\alpha = 4\pi R^3$(α is called the polarizability). (2) Find the induced surface charge distribution.*

Discussion: The system is shown in **Figure 5.4**. The applied external field can be written as

$$\vec{E}_{ext} = E_0\hat{z}.$$

Inside the conductor, there is an induced electric field that is opposite to the external field,

$$\vec{E}_{ind} = -E_0\hat{z}.$$

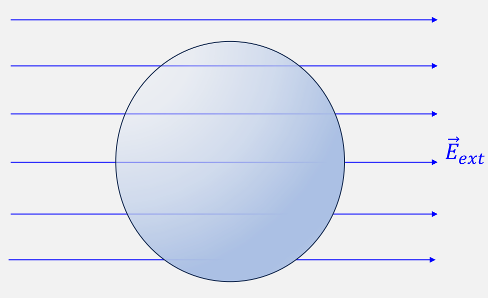


Fig. 5.4 An uniform electric field applied on a conductive sphere.

Outside the conductor, the induced electric field can be treated as a dipole field as shown by **Equation 5.12**. Thus, from the inside sphere, we have

$$\varphi_{ind}(r_- \to R, \theta) = E_0 z = E_0 R\cos\theta.$$

While at the outside sphere, the potential can be written as

$$\varphi_{ind}(r_+ \to R, \theta) = \frac{B}{R^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{R}}{R^3} = \frac{1}{4\pi\varepsilon_0}\frac{p}{R^2}\cos\theta.$$

At the surface of the sphere, $\varphi_{ind}(r_- \to R, \theta) = \varphi_{ind}(r_+ \to R, \theta)$, i.e.,

$$E_0 R\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{p}{R^2}\cos\theta.$$

Therefore,

$$p = 4\pi\varepsilon_0 R^3 E_0.$$

According to the definition of polarizability, $\vec{p} = \alpha\varepsilon_0\vec{E}_0$, we get

$$\alpha = 4\pi\varepsilon_0 R^3.$$

The total electric potential outside the conducting sphere can be written as (superposition of the potential generated by the uniform field and the field induced by the dipole on the conductor),

$$\varphi_{out}(\vec{r}) = -E_0 r\cos\theta + \frac{R^3 E_0}{r^2}\cos\theta.$$

According to **Equation 5.2**,

$$\sigma_s = \varepsilon_0 E_{out}(r = R) = -\varepsilon_0 \left.\frac{\partial E_{out}}{\partial r}\right|_{r=R} = 3\varepsilon_0 E_0\cos\theta.$$

Some interesting problems to think about:

- Consider the following surface change distribution of a charged conductor:
  (1) A conducting sphere
  (2) A conducting spheroid
  (3) A conducting disk
- Considering the following induced charge distributions:
  (1) A point charge in a cavity of a conductor
  (2) A point charge outside a conducting sphere
  (3) A dipole inside/outside a spherical conducting shell
- Induced charge distribution in a charged conductor:
  (1) A charged conducting sphere in a uniform external electric field
  (2) A charged conducting spheroid in a uniform external electric field

5.1.3 The force and torque on a conductor

When a conductor is placed in an external field, due to the induced charge distribution shown in **Figure 5.3**, there is a net electrostatic force acting on the conductor,

$$\vec{F} = \frac{1}{2}\iint_S \sigma(\vec{r}_s)\vec{E}_{out}(\vec{r}_s)\, dS' = \frac{1}{2\varepsilon_0}\iint_S \sigma(\vec{r}_s)^2\, \hat{n} dS' \,. \quad (5.16)$$

Similarly, there is a torque acting on the conductor,

$$\vec{N} = \frac{1}{2\varepsilon_0}\iint_S \sigma(\vec{r}_s)^2\, \vec{r}_s \times \hat{n} dS'. \quad (5.17)$$

5.2 Multiple Conductors and Capacitor

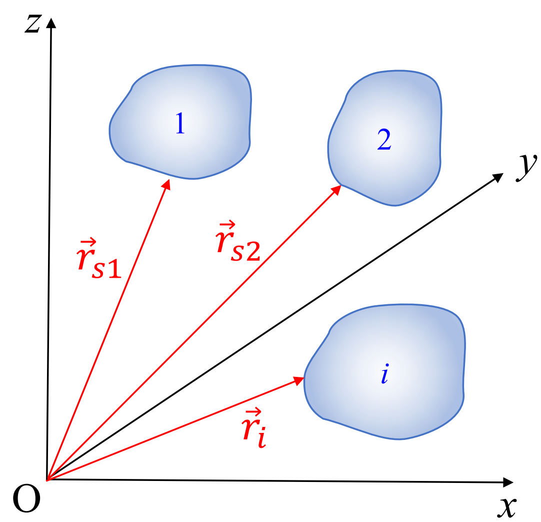


Fig. 5.5 An arrangement of *N* conductors.

When multiple charged conductors (total number of N) are distributed in space, the electric potential φ_i on the i-th conductor can be written as,

$$\varphi_i(\vec{r}_i) = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{N}\iint_{S_j}\frac{\sigma(\vec{r}_{sj})}{|\vec{r}_i - \vec{r}_{sj}|} dS'_j. \quad (5.18)$$

This equation illustrates how the potential at a conductor depends on the charge distributions $\sigma(\vec{r}_{sj})$ on all other conductors j. The potential from each conductor contributes to the total potential at the point of interest on the i-th conductor. Rearranging the equation allows us to incorporate the total charge q_j on each conductor

$$\varphi_i(\vec{r}_i) = \sum_{j=1}^{N} q_j \frac{1}{4\pi\varepsilon_0}\iint_{S_j}\frac{\sigma(\vec{r}_{sj})}{q_j|\vec{r}_i - \vec{r}_{sj}|} dS'_j = \sum_{j=1}^{N} q_j P_{ij}, \quad (5.19)$$

where P_{ij} is defined as

$$P_{ij} = \frac{1}{4\pi\varepsilon_0} \iint_{S_j} \frac{\sigma(\vec{r}_{sj})}{q_j|\vec{r}_i - \vec{r}_{sj}|} dS_j', \tag{5.20}$$

referred to as the coefficient of potential. Thus, the relationship between the potentials of the conductors and their charges can be summarized in matrix form,

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1N} \\ P_{21} & P_{22} & \cdots & P_{2N} \\ \vdots & \vdots & & \vdots \\ P_{N1} & P_{N2} & \dots & P_{N3} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}. \tag{5.21}$$

This matrix equation emphasizes how the potential on each conductor is influenced by the charge distributions on all other conductors. The physical interpretation of P_{ij} is that if a unit charge (1 C) is placed on the *j*-th conductor, the resulting potential difference between the *i*-th and *j*-th conductors will be equal to P_{ij}. Due to the reciprocal nature of electrostatics, we expect

$$P_{ij} = P_{ji}, \tag{5.22}$$

which indicates that the matrix P is symmetric. The inverse matric, $C = P^{-1}$, called the capacitance matrix, relates the charges and potentials,

$$\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \dots & C_{N3} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix}. \tag{5.23}$$

Here C_{ij} represents the coefficient of capacitance. Note that $C_{ij} = C_{ji}$, C_{ij} can be negative, and C_{ii} is called the self-capacitance of the conductor.

Self-Capacitance of a Single Conductor

To understand the self-capacitance of a single conductor, consider the total charge q on the conductor, which is given by,

$$q = \iint_S \sigma(\vec{r}_s)\, dS'. \tag{5.24}$$

The potential φ_s at the charge center of the conductor can be expressed as,

$$\varphi_s = \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\sigma(\vec{r}_s)}{|\vec{r}_s|} dS'. \tag{5.25}$$

From these, we can derive the self-capacitance C_{self},

$$C_{self} = \frac{q}{\varphi_s} = \frac{\iint_S \sigma(\vec{r}_s) dS'}{\frac{1}{4\pi\varepsilon_0} \iint_S \frac{\sigma(\vec{r}_s)}{|\vec{r}_s|} dS'}. \tag{5.26}$$

Examples of Self-Capacitance

(1) For a spherical conductor of radius R, $C_{self} = 4\pi\varepsilon_0 R$.

(2) For Earth, since $R \to \infty$, $C_{self} = \infty$, and the effective potential of the earth φ_{earth} is always zero regardless of the amount of charge placed upon it. This characteristic allows the Earth to be treated as a stable ground reference in electrical systems.

(3) For a conductor disk of radius a, $C_{self} = 8\varepsilon_0 a$.

Exploring the Capacitance Matrix

Examining the capacitance matrix further, for the *i*-th conductor, the surface charge density can be described by

$$\sigma_i = \varepsilon_0 \vec{E} \cdot \hat{n}_i = -\varepsilon_0 \nabla\phi \cdot \hat{n}_i, \tag{5.27}$$

where $\hat{n}_i$ is the unit vector normal to the surface S_i and ϕ is the potential near the surface S_i and $\phi \neq \varphi_i$. Therefore, the total charge q_i can be expressed as,

$$q_i = \oiint_{S_i} \sigma_i dS_i = -\varepsilon_0 \oiint_{S_i} \nabla\phi \cdot \hat{n}_i dS_i. \tag{5.28}$$

Since ϕ is the potential outside the conductors (but near the *i*-th conductor), it shall satisfy the Laplace equation,

$$\nabla^2\phi = 0, \tag{5.29}$$

the general solution can be expressed as,

$$\phi(\vec{r}) = \varphi_\infty + \sum_{j=1}^{N}(\varphi_j - \varphi_\infty) u_j(\vec{r}) = \sum_{j=1}^{N} \varphi_j u_j(\vec{r}), \tag{5.30}$$

where $u_j(\vec{r})$ are functions that satisfy the Laplace equation $\nabla^2 u_j = 0$, with $u_j(\vec{r}_{si}) = \delta_{ij}$. Using Gauss' theorem, we can establish the relationship

$$q_i = \sum_{j=1}^{N} C_{ij} \varphi_j, \tag{5.31}$$

with

$$C_{ij} = -\varepsilon_0 \oiint_{S_i} \nabla u_j \cdot \hat{n}_i dS_i. \tag{5.32}$$

Property of Capacitance Coefficients

The capacitance coefficients have several important properties,

$$\begin{cases} C_{ij} = C_{ji} \\ C_{ii} > 0 \\ C_{ij} < 0 \\ \sum_j C_{ij} \geq 0 \end{cases}. \tag{5.33}$$

For an enclosed system where no electric field lines extend to infinity (Figure xxx), we have,

$$\sum_j C_{ij} = 0. \tag{5.34}$$

Special Case: Two Conductors

Considering the case of two conductors, since $\varphi_i = \sum_{j=1}^{N} P_{ij}\, q_j$, we have

$$\begin{cases} \varphi_1 = P_{11}q_1 + P_{12}q_2 \\ \varphi_2 = P_{21}q_1 + P_{22}q_2 \end{cases}. \tag{5.36}$$

If we set $q_1 = -q_2 = q$ (a normal capacitor configuration), the potentials simplify to

$$\begin{cases} \varphi_1 = (P_{11} - P_{12})q \\ \varphi_2 = (P_{21} - P_{22})q \end{cases}. \tag{5.37}$$

The potential difference between the two conductors is given by

$$\Delta\varphi = \varphi_1 - \varphi_2 = [(P_{11} - P_{12}) - (P_{21} - P_{22})]q = (P_{11} + P_{22} - 2P_{12})q. \tag{5.38}$$

Thus, the capacitance C of the two-conductor capacitor can be expressed as,

$$C = \frac{q}{\Delta\varphi} = \frac{1}{P_{11}+P_{22}-2P_{12}}. \tag{5.39}$$

Since $P = C^{-1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}^{-1} = \frac{1}{C_{11}C_{22}-C_{12}^2}\begin{pmatrix} C_{22} & -C_{12} \\ -C_{12} & C_{11} \end{pmatrix}$, we have

$$C = \frac{C_{11}C_{22}-C_{12}^2}{C_{11}+C_{22}+2C_{12}} \tag{5.40}$$

If $C_{12} \ll C_{11}$ and C_{22} , then

$$\frac{1}{C} = \frac{1}{C_{11}} + \frac{1}{C_{22}}. \tag{5.41}$$

This implies that the capacitance of the two conductors can be approximated as the sum of their self-capacitances in series.

Core-Shell Structures

In configurations like parallel plate capacitors, spherical shells, and cylindrical shells, since $\sum_j C_{ij} = 0$, we find

$$C_{12} = C_{21} = -C_{11} = -C_{22} \tag{5.42}$$

In this case, there is only one independent coefficient, which we can denote as C_{11}. Therefore,

$$\begin{cases} q_1 = C_{11}(\varphi_1 - \varphi_2) \\ q_2 = -C_{11}(\varphi_1 - \varphi_2) = -q_1 \end{cases}. \tag{5.43}$$

The capacitance can then be expressed as,

$$C = \begin{cases} \varepsilon_0 \frac{A}{d}, & for\ parallel\ plates \\ \frac{4\pi\varepsilon_0 ab}{b-a}, & for\ phericsl\ core - dhell \\ \frac{2\pi\varepsilon_0 L}{\ln(\frac{b}{a})}, & for\ vylindrical\ co - axiel\ cable \end{cases}. \tag{5.44}$$

Example 5.2 *Find the capacitance of the two identical conducting spheres as shown in* ***Fig. 5.6****.*

For two identical conducting spheres with radius R, the coefficients of potential are given by

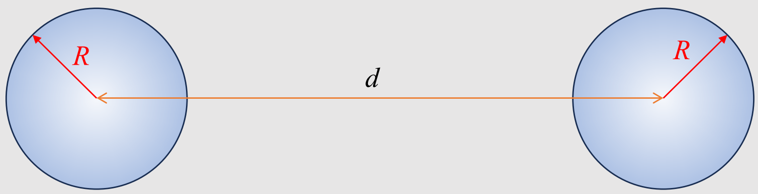


Fig. 5.6 Capacitance between two identical spheres.

$$P_{11} = P_{22} = \frac{1}{4\pi\varepsilon_0 R}$$

The potential of Sphere 1 generated by charge q_2 at Sphere 2 is

$$\varphi_{12} = \frac{q_2}{4\pi\varepsilon_0 d},$$

leading to,

$$P_{12} = \frac{1}{4\pi\varepsilon_0 d}.$$

According to **Equation 5.39**,

$$C = \frac{1}{P_{11} + P_{22} - 2P_{12}} = \frac{1}{\frac{2}{4\pi\varepsilon_0 R} - \frac{2}{4\pi\varepsilon_0 d}} = 2\pi\varepsilon_0 \frac{dR}{d - R}$$

If $d \gg R$, it is expected that $P_{12} \ll P_{11}$ and P_{22}. Therefore, the capacitance can be calculated as,

$$C \approx \frac{1}{P_{11}+P_{22}} = \frac{1}{\frac{1}{4\pi\varepsilon_0 R}+\frac{1}{4\pi\varepsilon_0 R}} = 2\pi\varepsilon_0 R.$$

This analysis provides a comprehensive view of multiple conductors and their capacitance properties, emphasizing the intricate relationships between charge, potential, and geometry in electrostatics.

Equivalent Circuit Approximation for Capacitors: When multiple conductors are placed close to each other, the concept of equivalent circuits can be employed to describe their electrical behavior, including capacitance. The arrangement of conductors and their interactions can be modeled by an equivalent circuit that simplifies the complex system into a more manageable form. For capacitive interactions between closely spaced conductors, as shown in **Figure 5.7**, the equivalent circuit may involve a network of capacitances representing the interactions between each pair of conductors. The total capacitance of the system is a combination of these individual capacitances, which can be calculated using series and parallel combinations depending on their configuration. It's important to note that the accuracy of such equivalent circuit models depends on the geometry, arrangement, and nature of the conductors. For more complex situations, numerical methods or simulations may be necessary to obtain precise results. As shown in the figure below, for multiple conductors arranged closely together, they can be approximated by an equivalent circuit of a capacitor network, where each capacitor in the network reflects the coupling effects between the conductors, allowing for an effective analysis of the overall capacitance and its implications in circuit design.

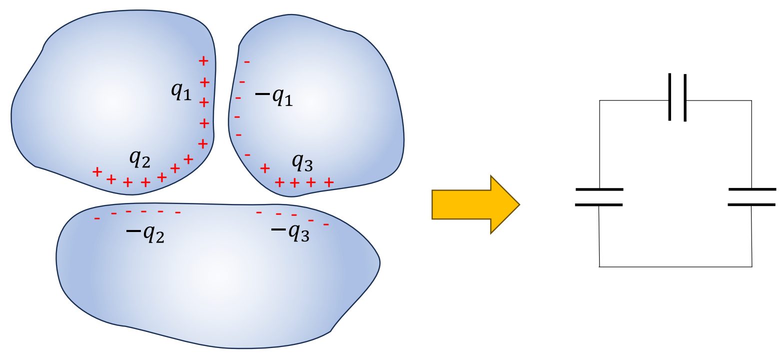


Fig. 5.7 Capacitance between two identical spheres.

The total electrostatic energy U_E of a system of conductors can be expressed using several formulations. According to **Equation 3.13**,

$$U_E = \frac{1}{2}\iiint_V \rho(\vec{r})\varphi(\vec{r})dV = \frac{1}{2}\sum_{i=1}^{N}\varphi_i \iint_{S_i} \sigma_i(\vec{r}_{si})dS_i' = \frac{1}{2}\sum_{i=1}^{N}\varphi_i q_i. \quad (5.45)$$

Here, q_i represents the total charge on conductor i, which can be derived from the potential φ_j of all conductors through the relationship $q_i = \sum_{j=1}^{N} C_{ij}\,\varphi_j$. Thus

$$U_E = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\varphi_j C_{ij}\,\varphi_i. \quad (5.46)$$

Alternatively, we can express the energy in terms of charges and the coefficients of potential,

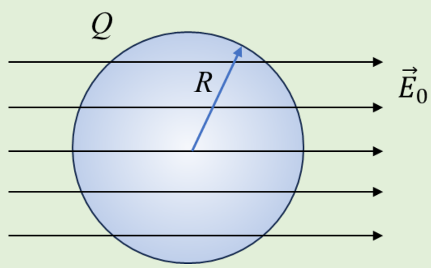
$$U_E = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} q_j P_{ij}\, q_i. \quad (5.47)$$

For a two-conductor system, this energy expression simplifies to,

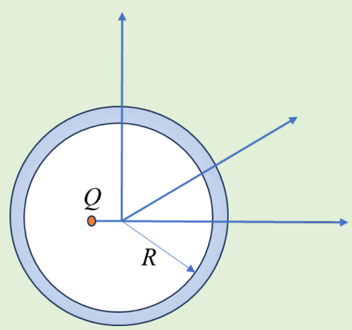
$$U_E = \frac{1}{2}q^2(P_{11} + P_{22} - 2P_{12}) = \frac{q^2}{2C} = \frac{1}{2}C(\varphi_1 - \varphi_2)^2 \quad (5.48)$$

In-class Activity

5-1. Find the electric potential of a charged conductor sphere in a uniform electric field.



5-2. A point charge Q is located at (-a, 0, 0) inside a spherical shell conductor as shown. How to obtain the surface charge density of the inner surface of the shell? What is the surface charge density of the outer surface?



5-3. Find self-capacitance of a uniformly charged disc.

5-4. Find self-capacitance of an isolated conducting disc.