Chapter 11
Magnetic Materials

11.1 Magnetization

All materials are made of atoms and molecules, and the electrons of these particles shall orbit around their nucleus according to the classic view, which, from **Section 10.1**, can be treated as magnetic dipoles. If all the atoms or molecules are in a regular lattice and the corresponding electrons all orbiting along the same direction, there will be no net magnetic moment existing in the material as shown in **Figure 11.1A**: in any dashed green boxed, effectively all the molecular currents are cancelled and there shall be no net magnetic dipole moments in the material. However, for a material with defects, such as a vacancy shown in **Figure 11.1B**, as indicated by the blue dashed arrow, there will be a magnetic dipole presented in the materials. Thus, for most materials that are not perfect, essentially, due to the local none-cancelled molecular current, the material could have a bulk distribution of magnetic dipoles. Similar to the definition of polarization, we define magnetization $\vec{M}$ of a material as,

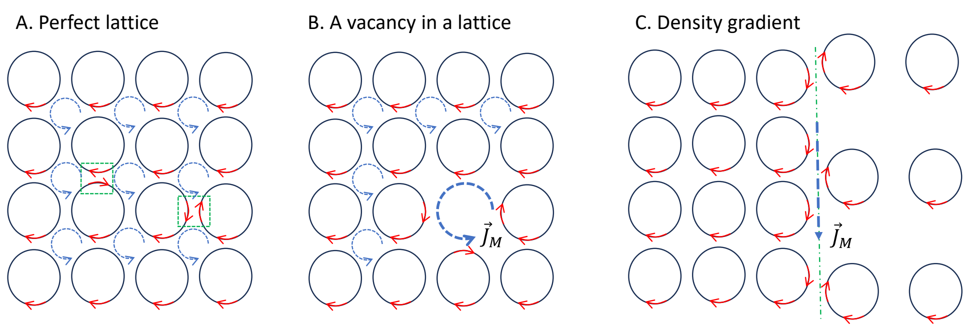

Fig. 11.1 Molecular current in a material: (A) perfect lattice; (B) a lattice with a vacancy; and (C) lattices with two magnetic dipole densities.

$$\vec{M} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i \vec{m}_i, \tag{11.1}$$

i.e., $\vec{M}$ is defined as magnetic dipole density. Let's assume that the vector function $\vec{M}(x, y, z)$ is known for a material, thought the formula could be complicated. If the magnetic dipoles are uniformly distributed in the materials, there will be not net local current $\vec{J}_M(\vec{r})$ presented inside the material, as shown in the green dashed boxes of **Figure 11.1A**. However, if there is a defect (**Figure 11.1B**) or non-uniform distribution of the magnetic dipoles (**Figure 11.1C**), at the defect location or at the boundary between two different densities, a net current can exist $\vec{J}_M$.

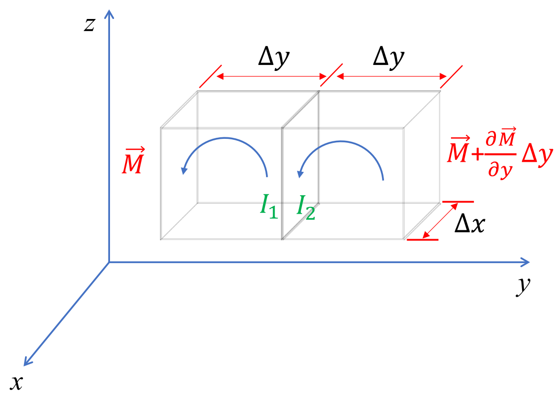

Fig. 11.2 The induced local current at two adjacent volumes along *y*-direction.

Let's consider the relationship between the net local current $\vec{J}_M(\vec{r})$ and the distribution of $\vec{M}(x,y,z)$. As illustrated in **Figure 11**, to obtain an $\vec{J}_M(\vec{r})$, the $\vec{M}(\vec{r})$ shall have a change at $\vec{r}$ location. Let's first assume such a change is in *y*-direction, i.e.,

$$\vec{M} = \vec{M}(x,y,z) + \frac{\partial \vec{M}}{\partial y}\Delta y. \tag{11.2}$$

Consider two small adjacent volumes in *y*-direction as shown in **Figure 12**. In the *y*-*z* plane, there are two loop currents formed due to the local magnetization nonuniformity: at the boundary between these two volumes, I_1 is along the positive *z*-direction while I_2 points to the negative *z*-direction. Clear, in this case, only M_x contributes to these two currents, so that

$$M_x \Delta x \Delta y \Delta z = I_1 \Delta y \Delta z, \tag{11.3}$$

for the left volume. And for the right volume, one has,

$$(M_x + \frac{\partial M_x}{\partial y}\Delta y)\Delta x \Delta y \Delta z = I_2 \Delta y \Delta z. \tag{11.4}$$

The net current at the boundary can be written as,

$$I_1 - I_2 = -\frac{\partial M_x}{\partial y}\Delta y \Delta x, \tag{11.5}$$

thus the net local current density along *z*-direction can be written as,

$$(J_M)_z = \frac{I_1 - I_2}{\Delta y \Delta x} = -\frac{\partial M_x}{\partial y}. \tag{11.6}$$

Similarly, if one considers two adjacent small volumes along the *x*-direction, another contribution to the z-direction current density can be found,

$$(J_M)_z = \frac{\partial M_y}{\partial x}. \tag{11.7}$$

Thus, the total contribution to the z-direction local current density is,

$$(J_M)_z = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}. \tag{11.8}$$

The expression on the right-hand side of **Equation 11.8** is the z-component of the curl of $\vec{M}$. Similar argument can be used to find $(J_M)_x$ and $(J_M)_y$, and finally we obtain,

$$\vec{J}_M = \nabla \times \vec{M}. \tag{11.9}$$

The bounded local current is determined by the curl of the magnetization $\vec{M}$.

11.2 The Magnetic Field Produced by Magnetization

In a small volume of a magnetic material, the total magnetic dipole moments can be written as,

$$\Delta\vec{m} = \vec{M}\Delta V'. \tag{11.10}$$

Therefore, the vector potential produced by these magnetic dipole moments can be written as,

$$\Delta\vec{A} = \frac{\mu_0}{4\pi}\frac{\Delta\vec{m}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}, \tag{11.11}$$

The overall vector potential $\vec{A}$ can be expressed as,

$$\vec{A} = \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{M}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dV' = \frac{\mu_0}{4\pi}\iiint_V \vec{M}\times\nabla'\frac{1}{|\vec{r}-\vec{r}'|}dV'. \tag{11.12}$$

Using the following identities,

$$\begin{cases} \nabla\times\left(\varphi\vec{F}\right) = (\nabla\varphi)\times\vec{F} + \varphi\nabla\times\vec{F} \\ \iiint_V \nabla\times\vec{F}dV' = \oiint_S \hat{n}\times\vec{F}\,dS' \end{cases}, \tag{11.13}$$

We have

$$\vec{A} = \frac{\mu_0}{4\pi}\iiint_V \frac{\nabla'\times\vec{M}}{|\vec{r}-\vec{r}'|}dV' + \frac{\mu_0}{4\pi}\oiint_S \frac{\vec{M}\times\hat{n}}{|\vec{r}-\vec{r}'|}dS'. \tag{11.14}$$

Since $\nabla'\times\vec{M} = \vec{J}_M(\vec{r}')$ and $\vec{M}\times\hat{n} = \vec{K}_M(\vec{r}_s)$, **Equation 11.14** can be rewritten as,

$$\vec{A} = \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{J}_M(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' + \frac{\mu_0}{4\pi}\oiint_S \frac{\vec{K}_M(\vec{r}_s)}{|\vec{r}-\vec{r}'|}dS'. \tag{11.15}$$

If we compare the expression in **Equation 11.15** to the expression for electrostatic potential, the first term in the right-hand side represents a bulk contribution, and the second term is a result from a surface contribution.

The magnetic field generated by this magnetized material can be written as,

$$\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint_V \nabla \times \left[\frac{\vec{M}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right] dV'. \quad (11.16)$$

Using the identity, $\nabla \times (\vec{F}\times\vec{G}) = (\nabla\cdot\vec{G})\vec{F} - (\nabla\cdot\vec{F})\vec{G} + (\vec{G}\cdot\nabla)\vec{F} - (\vec{F}\cdot\nabla)\vec{G}$, we have,

$$\nabla \times \left[\vec{M}\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right] = \vec{M}\nabla\cdot\left[\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right] - (\vec{M}\cdot\nabla)\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}. \quad (11.17)$$

Here we use the relationship $\nabla\cdot\vec{M} = 0$ inside a bulk magnetic material. Therefore, the magnetic field consists of two parts, $\vec{B}(\vec{r}) = \vec{B}_I(\vec{r}) + \vec{B}_{II}(\vec{r})$, with

$$\vec{B}_I(\vec{r}) = \frac{\mu_0}{4\pi}\iiint_V \vec{M}\nabla\cdot\left[\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right] dV', \quad (11.18)$$

$$\vec{B}_{II}(\vec{r}) = -\frac{\mu_0}{4\pi}\iiint_V (\vec{M}\cdot\nabla)\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} dV'. \quad (11.19)$$

For $\vec{B}_I(\vec{r})$, $\nabla\cdot\left[\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right] = 4\pi\delta(\vec{r}-\vec{r}')$, thus,

$$\vec{B}_I(\vec{r}) = \mu_0\vec{M}. \quad (11.20)$$

For $\vec{B}_{II}(\vec{r})$, since $\nabla\left[\vec{M}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right] = (\vec{M}\cdot\nabla)\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} + \vec{M}\times\left(\nabla\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$, the second term on the right-hand side is zero, thus,

$$\vec{B}_{II}(\vec{r}) = -\mu_0\nabla\frac{1}{4\pi}\iiint_V \vec{M}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} dV'. \quad (11.21)$$

Therefore, we can define a magnetic scalar potential φ_B,

$$\varphi_B = \frac{1}{4\pi}\iiint_V \vec{M}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} dV', \quad (11.22)$$

And

$$\vec{B}_{II}(\vec{r}) = -\mu_0\nabla\varphi_B. \quad (11.23)$$

Based on **Equations 11.20** and **11.23**, we have

$$\vec{B}(\vec{r}) = -\mu_0\nabla\varphi_B + \mu_0\vec{M}. \quad (11.24)$$

11.3 Magnetic Scalar Potential

Since $\vec{M}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \vec{M}\cdot\nabla'\frac{1}{|\vec{r}-\vec{r}'|} = \nabla'\cdot\frac{\vec{M}}{|\vec{r}-\vec{r}'|} - \frac{1}{|\vec{r}-\vec{r}'|}\nabla'\cdot\vec{M}$, thus

$$\varphi_B = \frac{1}{4\pi}\oiint_S \frac{\vec{M}\cdot\hat{n}}{|\vec{r}-\vec{r}'|}dS' - \frac{1}{4\pi}\iiint_V \frac{\nabla'\cdot\vec{M}}{|\vec{r}-\vec{r}'|}dV', \tag{11.25}$$

So we can define the magnetic charge density,

$$\rho_M(\vec{r}') = -\nabla'\cdot\vec{M}, \tag{11.26}$$

and surface magnetic charge density,

$$\sigma_M(\vec{r}_S) = \vec{M}(\vec{r}_S)\cdot\hat{n}, \tag{11.27}$$

By considering both fields, we have,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint_V \rho_M(\vec{r}')\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}dV' + \frac{\mu_0}{4\pi}\oiint_S \sigma_M(\vec{r}_S)\frac{\vec{r}-\vec{r}_S}{|\vec{r}-\vec{r}_S|^3}dS' + \mu_0\vec{M}. \tag{11.28}$$

11.4 Magnetic Intensity

If the material also has a current density $\vec{J}(\vec{r})$, then

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dV' - \mu_0\nabla\varphi_B(\vec{r}) + \mu_0\vec{M}(\vec{r}). \tag{11.29}$$

To make the expression more concise, we can define the magnetic intensity $\vec{H}(\vec{r})$, with

$$\vec{H}(\vec{r}) = \frac{1}{\mu_0}\vec{B}(\vec{r}) - \vec{M}(\vec{r}), \tag{11.30}$$

so that

$$\vec{H}(\vec{r}) = \frac{1}{4\pi}\iiint_V \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dV' - \nabla\varphi_B(\vec{r}). \tag{11.31}$$

Since $\nabla\cdot\vec{B}(\vec{r}) = 0$ and $\nabla\times\vec{B}(\vec{r}) = \mu_0\vec{J}_{total}$ with $\vec{J}_{total} = \vec{J} + \vec{J}_M$, i.e.,

$$\nabla\times\vec{B}(\vec{r}) = \mu_0(\vec{J} + \vec{J}_M). \tag{11.32}$$

Then,

$$\nabla\times\vec{H}(\vec{r}) = \nabla\times\left[\frac{1}{\mu_0}\vec{B} - \vec{M}\right] = \vec{J}. \tag{11.33}$$

Only free current density is shown in the relationship. Therefore,

$$\iint_S \nabla\times\vec{H}(\vec{r})\cdot\hat{n}dS' = \oint_L \vec{H}(\vec{r})\cdot d\vec{l} = \iint_S \vec{J}\cdot\hat{n}dS'. \tag{11.34}$$

Thus,

$$\oint_L \vec{H}(\vec{r})\cdot d\vec{l} = \sum I. \tag{11.35}$$

This is the Ampere's law.

The constitutive relationship,

$$\vec{M} = \chi_M \vec{H}. \tag{11.36}$$

χ_M is called the magnetic susceptibility. When $\chi_M > 0$, the material is called paramagnetic material; for $\chi_M < 0$, the material is called diamagnetic material. This implies that,

$$\vec{B} = \mu \vec{H}, \tag{11.37}$$

where $\mu = \mu_0(1 + \chi_M)$.

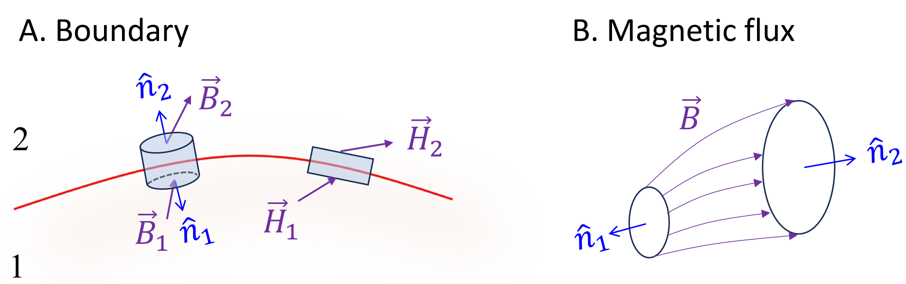

Fig. 11.3 (A) The boundary for magnetic fields and (B) the magnetic flux.

11.5 Boundary Conditions

Since $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{H} = \vec{J}$, at the boundary as shown in **Figure 11.3**, using both the Gaussian and Stokes theorems, one can obtain the following boundary conditions,

$$\begin{cases} \hat{n}_2 \cdot (\vec{B}_2 - \vec{B}_1) = 0 \\ \hat{n}_2 \times (\vec{H}_2 - \vec{H}_1) = \vec{K}(\vec{r}_s) \end{cases}, \tag{10.38}$$

here $\vec{K}(\vec{r}_s)$ is the surface current density at the boundary. Also, according to the Gauss's law of $\vec{B}$, the magnetic flux should be continuum,

$$\iiint_V \nabla \cdot \vec{B} dV' = \oiint_S \vec{B} \cdot \hat{n} dS'$$

$$= \iint_{S_2} \vec{B} \cdot \hat{n} dS' - \iint_{S_1} \vec{B} \cdot \hat{n} dS' = \Phi(S_2) - \Phi(S_1). \tag{10.39}$$

Since $\vec{B} = \mu_0(\vec{H} + \vec{M})$, $\nabla \cdot \vec{B} = \mu_0(\nabla \cdot \vec{H} + \nabla \cdot \vec{M})$, therefore,

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}. \tag{10.40}$$

Thus,

$$\iiint_V \nabla \cdot \vec{H} dV' = -\iiint_V \nabla \cdot \vec{M} dV'$$

$$\iint_{S_2} \vec{H} \cdot \hat{n} dS' - \iint_{S_1} \vec{H} \cdot \hat{n} dS' = \iiint_V \rho_M(\vec{r}')\, dV'. \tag{10.41}$$

Boundary value problem with magnetic material:

Since $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{H} = 0$, then $\vec{H} = -\nabla\varphi_B(\vec{r})$. Consider two types of magnetic materials,

(1) Linear or approximately linear magnetic materials: $\vec{B} = \mu\vec{H}$.

(2) A uniformly magnetized piece of material: $\nabla \cdot \vec{M} = 0$.

In both case, $\nabla \cdot \vec{H} = 0$, so that,

$$\nabla^2\varphi_B(\vec{r}) = 0. \tag{11.42}$$

Then it becomes the problem to solve for a Laplace equation.