Chapter 10
Magnetic Multipole, Force, and Energy

10.1 Magnetic Multipole Expansion

For a finite volume of current distribution as shown in **Figure 10.1**, with the source radius $r' < R$, where R is the maximum source radius of the system, the vector potential can be written as,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} dV'. \tag{10.1}$$

Like the electrostatic potential, for $r \gg R$, the vector potential $\vec{A}(\vec{r})$ can be expanded into a multipole format,

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} - \vec{r}' \cdot \nabla \frac{1}{r} + \frac{1}{2}(\vec{r}' \cdot \nabla)^2 \frac{1}{r} + \cdots \tag{10.2}$$

Or,

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos\theta) \tag{10.3}$$

with $\cos\theta = \hat{r} \cdot \hat{r}'$. Therefore,

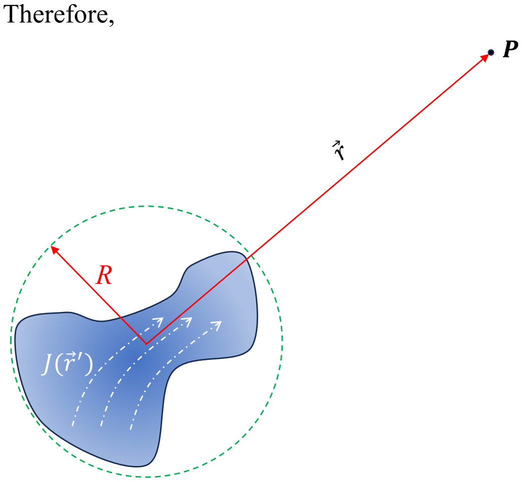

Fig. 10.1 Current source and far field configuration.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left\{\begin{matrix} \frac{1}{r}\iiint_V \vec{J}(\vec{r}')dV' + \frac{1}{r^2}\iiint_V (\hat{r}\cdot\hat{r}')\vec{J}(\vec{r}')dV' \\ +\frac{1}{r^3}\iiint_V \left[\frac{3}{2}(\hat{r}\cdot\hat{r}')^2 - \frac{1}{2}r'^2\right]\vec{J}(\vec{r}')dV' + \cdots \end{matrix}\right\}. \quad (10.4)$$

Or

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\iiint_V r'^l P_l(\cos\theta)\vec{J}(\vec{r}')dV'. \quad (10.5)$$

The first term in **Equations 10.4** or **10.5** is from monopole contribution; the second term is from a dipole; the third term is from a quadrupole; and so on.

❖ Let's look at the first term of **Equation 10.4**: $\iiint_V \vec{J}(\vec{r}')dV'$

Using the following identity,

$$\nabla'\cdot\left[\left(\vec{C}\cdot\vec{r}'\right)\vec{J}(\vec{r}')\right] = \vec{C}\cdot\vec{J}(\vec{r}') + \left(\vec{C}\cdot\vec{r}'\right)\left[\nabla'\cdot\vec{J}(\vec{r}')\right], \quad (10.6)$$

where $\vec{C}$ is an arbitrary constant vector. And for magnetostatics, we also have $\nabla'\cdot\vec{J}(\vec{r}') = 0$, thus

$$\nabla'\cdot\left[\left(\vec{C}\cdot\vec{r}'\right)\vec{J}(\vec{r}')\right] = \vec{C}\cdot\vec{J}(\vec{r}'), \quad (10.7)$$

Performing a volume integration of the object shown in **Figure 10.1**, we have

$$\iiint_V \nabla'\cdot\left[\left(\vec{C}\cdot\vec{r}'\right)\vec{J}(\vec{r}')\right]dV' = \iiint_V \vec{C}\cdot\vec{J}(\vec{r}')\,dV', \quad (10.8)$$

According to Gauss's theorem, the left side of **Equation 10.8** can be written as,

$$\iiint_V \nabla'\cdot\left[\left(\vec{C}\cdot\vec{r}'\right)\vec{J}(\vec{r}')\right]dV' = \oiint_S \left(\vec{C}\cdot\vec{r}'\right)\vec{J}(\vec{r}')\cdot d\vec{S}'. \quad (10.9)$$

Here S is the enclosed surface that surrounds the current source shown in **Figure 10.1** and can be chosen arbitrarily as long as the surface encloses the current source. Thus, if the radius of the surface S is large enough, shown as the dashed sphere in **Figure 10.1**, the current density $\vec{J}(\vec{r}')$ on the boundary surface is zero. Thus,

$$\vec{C}\cdot\iiint_V \vec{J}(\vec{r}')\,dV' = 0. \quad (10.10)$$

Since $\vec{C}$ is an arbitrary constant vector and **Equation 10.10** is valid for any $\vec{C}$, thus

$$\iiint_V \vec{J}(\vec{r}')\,dV' = 0. \quad (10.11)$$

Therefore, the monopole contribution to the magnetic vector potential $\vec{A}(\vec{r})$ is zero. This result is consistent with the fact that there is no magnetic monopole for electromagnetism.

❖ Let's look at the second term of **Equation 10.4**: $\iiint_V (\hat{r}\cdot\hat{r}')\vec{J}(\vec{r}')dV'$

Let's looking at the following two identities,

$$\vec{C}\cdot\{\hat{r}\times[\vec{J}(\vec{r}')\times\vec{r}']\} = [\vec{C}\cdot\vec{J}(\vec{r}')](\hat{r}\cdot\vec{r}') - (\vec{C}\cdot\vec{r}')[\hat{r}\cdot\vec{J}(\vec{r}')], \quad (10.12)$$

$$\nabla'\cdot[(\vec{C}\cdot\vec{r}')(\hat{r}\cdot\vec{r}')\vec{J}(\vec{r}')] = [\vec{C}\cdot\vec{J}(\vec{r}')](\hat{r}\cdot\vec{r}') + (\vec{C}\cdot\vec{r}')[\hat{r}\cdot\vec{J}(\vec{r}')] + (\vec{C}\cdot\vec{r}')(\hat{r}\cdot\vec{r}')[\nabla'\cdot\vec{J}(\vec{r}')]. \quad (10.13)$$

The 3rd term on the right side of **Equation 10.13** is zero since $\nabla'\cdot\vec{J}(\vec{r}') = 0$. Adding **Equations 10.12** and **10.13** together, we have

$$[\vec{C}\cdot\vec{J}(\vec{r}')](\hat{r}\cdot\vec{r}') = \frac{1}{2}\vec{C}\cdot\{\hat{r}\times[\vec{J}(\vec{r}')\times\vec{r}']\} + \frac{1}{2}\nabla'\cdot[(\vec{C}\cdot\vec{r}')(\hat{r}\cdot\vec{r}')\vec{J}(\vec{r}')], \quad (10.14)$$

Therefore,

$$\vec{C}\cdot\iiint_V (\hat{r}\cdot\hat{r}')\vec{J}(\vec{r}')dV' = \frac{1}{2}\vec{C}\cdot\iiint_V \hat{r}\times[\vec{J}(\vec{r}')\times\vec{r}']dV' + \frac{1}{2}\iiint_V \nabla'\cdot[(\vec{C}\cdot\vec{r}')(\hat{r}\cdot\vec{r}')\vec{J}(\vec{r}')]dV'$$

$$= \frac{1}{2}\vec{C}\cdot\iiint_V \hat{r}\times[\vec{J}(\vec{r}')\times\vec{r}']dV' + \frac{1}{2}\oiint_S (\vec{C}\cdot\vec{r}')(\hat{r}\cdot\vec{r}')\vec{J}(\vec{r}')\cdot d\vec{S}'. \quad (10.15)$$

The second term on the right-hand side of **Equation 10.15** is zero according to the same argument for **Equation 10.9**. Thus

$$\vec{C}\cdot\iiint_V (\hat{r}\cdot\hat{r}')\vec{J}(\vec{r}')dV' = \frac{1}{2}\vec{C}\cdot\iiint_V \hat{r}\times[\vec{J}(\vec{r}')\times\vec{r}']dV'$$

Since $\vec{C}$ is an arbitrary constant vector, therefore,

$$\iiint_V (\hat{r}\cdot\hat{r}')\vec{J}(\vec{r}')dV' = \frac{\hat{r}}{2}\times\iiint_V \vec{J}(\vec{r}')\times\vec{r}'dV' = \left[\frac{1}{2}\iiint_V \vec{r}'\times\vec{J}(\vec{r}')dV'\right]\times\hat{r}. \quad (10.16)$$

Let's define the magnetic dipole moment $\vec{m}$ as

$$\vec{m} = \frac{1}{2}\iiint_V \vec{r}'\times\vec{J}(\vec{r}')dV', \quad (10.17)$$

we have

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\vec{m}\times\vec{r}. \quad (10.18)$$

Let's consider a very special case, a current loop as shown in **Figure 10.2**. The vector potential $\vec{A}(\vec{r})$ at **P** location can be written as,

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint_L \frac{d\vec{l}}{|\vec{r}-\vec{r}'|} = \frac{\mu_0 I}{4\pi}\frac{1}{r^{l+1}}\sum_{l=0}^{\infty}\oint_L r'^l P_l(\cos\theta)d\vec{r}' =$$

$$\frac{\mu_0}{4\pi}\left\{\frac{1}{r}\oint_L d\vec{r}' + \frac{1}{r^2}\oint_L (\hat{r}\cdot\hat{r}')d\vec{r}' + \frac{1}{r^3}\oint_L \left[\frac{3}{2}(\hat{r}\cdot\hat{r}')^2 - \frac{1}{2}r'^2\right]d\vec{r}' + \cdots\right\}. \quad (10.19)$$

The first term, $\oint_L d\vec{r}' = 0$. For the second term, we have

$$\vec{C}\cdot\oint_L (\hat{r}\cdot\hat{r}')d\vec{r}' = \oint_L (\hat{r}\cdot\hat{r}')\vec{C}\cdot d\vec{r}'$$

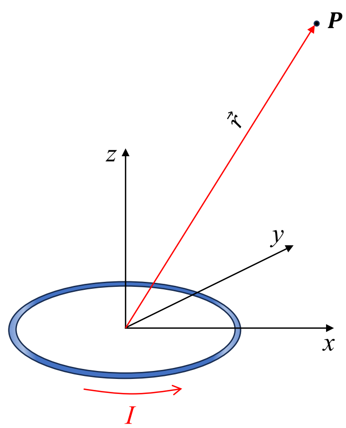

Fig. 10.2 A current loop.

$$= \iint_S \nabla' \times \left[(\hat{r}\cdot\hat{r}')\vec{C}\right]\cdot d\vec{S}'. \quad (10.20)$$

Since $\nabla' \times \left[(\hat{r}\cdot\hat{r}')\vec{C}\right] = [\nabla'(\hat{r}\cdot\hat{r}')]\times\vec{C} = \hat{r}\times\vec{C}$, thus

$$\vec{C}\cdot\oint_L (\hat{r}\cdot\hat{r}')d\vec{r}' = \iint_S \left(\hat{r}\times\vec{C}\right)\cdot d\vec{S}' = \left(\hat{r}\times\vec{C}\right)\cdot\iint_S d\vec{S}'$$

$$= \left(\hat{r}\times\vec{C}\right)\cdot\vec{a} = (\vec{a}\times\hat{r})\cdot\vec{C}, \quad (10.21)$$

where $\vec{a}$ is the area vector of the current loop. Therefore,

$$\oint_L (\hat{r}\cdot\hat{r}')d\vec{r}' = \vec{a}\times\hat{r}. \quad (10.22)$$

Finally, according to **Equation 10.19**, we have,

$$\vec{A}_{loop}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\vec{a}\times\vec{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}. \quad (10.23)$$

❖ The magnetic field of a magnetic dipole

$$\vec{B}(\vec{r}) = \nabla\times\vec{A}_{dipole}(\vec{r}) = \nabla\times\left(\frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}\right) = \frac{\mu_0}{4\pi}\left[\vec{m}\cdot\left(\nabla\cdot\frac{\vec{r}}{r^3}\right) - (\vec{m}\cdot\nabla)\frac{\vec{r}}{r^3}\right]. \quad (10.24)$$

Since $\nabla\cdot\frac{\vec{r}}{r^3} = -4\pi\delta(\vec{r})$, at $\vec{r}\neq 0$, only the second term in **Equation 10.24** is valid,

$$\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}(\vec{m}\cdot\nabla)\frac{\vec{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{3\hat{r}(\hat{r}\cdot\vec{m})-\vec{m}}{r^3}. \quad (10.25)$$

Compare this equation to the electric field of an electric dipole, $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{3\hat{r}(\hat{r}\cdot\vec{p})-\vec{p}}{r^3}$, they have very similar expression.

❖ Orbital and spin magnetic dipole moment

From a classical point of view, the electrons of a molecule or an atom can be treated as charged particles orbiting with a specific trajectory around the nucleus. If the charge is q_k, with a velocity $\vec{v}_k = \frac{d\vec{r}_k}{dt}$ and a mass m_k, the current density can be written as

$$\vec{J}(\vec{r}) = \sum_{k=1}^{N} q_k \vec{v}_k \delta(\vec{r} - \vec{r}_k). \tag{10.26}$$

Thus,

$$\vec{m}_L = \frac{1}{2}\iiint_V \vec{r}' \times \vec{J}(\vec{r}')dV' = \frac{1}{2}\sum_{k=1}^{N} q_k \vec{r}_k \times \vec{v}_k = \sum_{k=1}^{N} \frac{q_k}{2m_k}\vec{L}_k. \tag{10.27}$$

Here $\vec{L}_k = m_k \vec{r}_k \times \vec{v}_k$ is the angular momentum of each orbiting particle. If all the particle has the same charge q and mass m, we have

$$\vec{m}_L = \frac{q}{2m}\sum_{k=1}^{N} m\, \vec{r}_k \times \vec{v}_k = \frac{q}{2m}\vec{L}, \tag{10.28}$$

here $\vec{L}$ is the total orbital angular momentum of the molecule or atom.

For a spin angular momentum $\vec{s}$, it has a magnetic moment,

$$\vec{m}_s = g\frac{q}{m}\vec{s}, \tag{10.29}$$

where g is called the g -factor.

10.2 Magnetic Force and Torque

Based on **Section 9.12**, the magnetic force $\vec{F}_B$ and torque $\vec{N}_B$ acting on a current source in a magnetic field $\vec{B}$ can be expressed as,

$$\vec{F}_B = \iiint_V \vec{J}(\vec{r}') \times \vec{B}(\vec{r}')\, dV', \tag{9.18}$$

$$\vec{N}_B = \iiint_V \vec{r}' \times \left[\vec{J}(\vec{r}') \times \vec{B}(\vec{r}')\right] dV'. \tag{9.19}$$

The magnetic force on a current carrying wire I in a magnetic field can be written as

$$\vec{F}_B = I\int_L d\vec{l}' \times \vec{B}(\vec{r}'). \tag{10.30}$$

The force on an object with a surface current density $\vec{K}(\vec{r}_s)$ is expressed as,

$$\vec{F}_B = \iint_S \vec{K}(\vec{r}_s) \times \vec{B}(\vec{r}_s)\, dS'. \tag{10.31}$$

And for moving charges with $\vec{J}(\vec{r}) = \sum_{k=1}^{N} q_k \vec{v}_k \delta(\vec{r} - \vec{r}_k)$ as shown in **Equation 10.26**, one has,

$$\vec{F}_B = \sum_{k=1}^{N} q_k \vec{v}_k \times \vec{B}(\vec{r}_k). \tag{10.32}$$

❖ The magnetic force between two current carrying objects

As shown in **Figure 10.3**, for two current carrying objects 1 and 2, Object 1 will generate a magnetic field $\vec{B}_1(\vec{r})$ at Object 2, therefore, a magnetic force $\vec{F}_2(\vec{r})$ will be produced on Object 2. The magnetic field produced at $\vec{r}$ can be written as,

$$\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi}\iiint_{V_1}\frac{\vec{J}_1(\vec{r}\prime)\times(\vec{r}-\vec{r}\prime)}{|\vec{r}-\vec{r}\prime|^3}dV'. \qquad (10.32)$$

Thus the force on Object 2 can be written as,

$$\vec{F}_2 = \iiint_{V_2}\vec{J}_2(\vec{r})\times\vec{B}_1(\vec{r})\,dV = \frac{\mu_0}{4\pi}\iiint_{V_2}dV\vec{J}_2(\vec{r})\times\iiint_{V_1}\frac{\vec{J}_1(\vec{r}\prime)\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}\prime|^3}dV'$$

$$= \frac{\mu_0}{4\pi}\iiint_{V_2}dV\iiint_{V_1}dV'\frac{\vec{J}_1(\vec{r}\prime)[\vec{J}_2(\vec{r})\cdot(\vec{r}-\vec{r}\prime)]-[\vec{J}_1(\vec{r}\prime)\cdot\vec{J}_2(\vec{r})](\vec{r}-\vec{r}\prime)}{|\vec{r}-\vec{r}\prime|^3}. \qquad (10.34)$$

Since

$$\iiint_{V_2}\nabla\cdot\frac{\vec{J}_2(\vec{r})}{|\vec{r}-\vec{r}\prime|}dV = \iiint_{V_2}\frac{\nabla\cdot\vec{J}_2(\vec{r})}{|\vec{r}-\vec{r}\prime|}dV + \iiint_{V_2}\vec{J}_2(\vec{r})\cdot\nabla\frac{1}{|\vec{r}-\vec{r}\prime|}dV,$$

and $\nabla\cdot\vec{J}_2(\vec{r}) = 0$, $\nabla\frac{1}{|\vec{r}-\vec{r}\prime|} = -\frac{\vec{r}-\vec{r}\prime}{|\vec{r}-\vec{r}\prime|^3}$, $\iiint_{V_2}\nabla\cdot\frac{\vec{J}_2(\vec{r})}{|\vec{r}-\vec{r}\prime|}dV = \oiint_{S_2}\frac{\vec{J}_2(\vec{r})}{|\vec{r}-\vec{r}\prime|}\cdot\hat{n}dS' = 0$, thus

$$\vec{F}_2 = -\frac{\mu_0}{4\pi}\iiint_{V_2}dV\iiint_{V_1}dV'\vec{J}_1(\vec{r}')\cdot\vec{J}_2(\vec{r})\frac{\vec{r}-\vec{r}\prime}{|\vec{r}-\vec{r}\prime|^3}. \qquad (10.35)$$

Based on **Equation 10.35**, we also expect that the magnetic force on Object 1 due to the magnetic field produced by Object 2 can be written as,

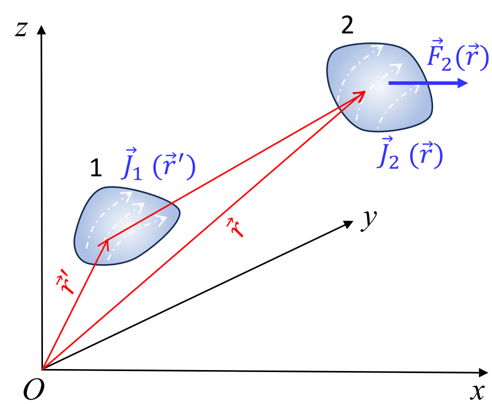

Fig. 10.3 The magnetic force between two current carrying objects.

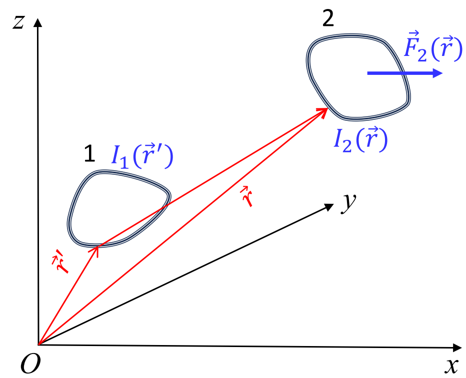

Fig. 10.4 The magnetic force between two current carrying loops.

$$\vec{F}_1 = \frac{\mu_0}{4\pi}\iiint_{V_2} dV \iiint_{V_1} dV' \vec{J}_1(\vec{r}') \cdot \vec{J}_2(\vec{r}) \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}. \tag{10.36}$$

Therefore,

$$\vec{F}_2 = -\vec{F}_1, \tag{10.37}$$

which satisfies Newton's third law.

Comparing **Equation 10.36** to the electrostatic force, they have very similar form,

$$\vec{F}_2 = \frac{1}{4\pi\varepsilon_0}\iiint_{V_2} dV \iiint_{V_1} dV' \rho_1(\vec{r}') \cdot \rho_2(\vec{r}) \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}. \tag{10.38}$$

❖ The magnetic force between two current carrying loops

As shown in **Figure 10.4**, Loop 1 will generate a magnetic field $\vec{B}_1(\vec{r})$ on a small section on Loop 2, therefore can produce a small force $d\vec{F}_2$,

$$d\vec{F}_2 = I_2\left[d\vec{l}_2 \times \vec{B}_1(\vec{r})\right], \tag{10.39}$$

the total force $\vec{F}_2$ acting on Loop 2 can be written as,

$$\vec{F}_2 = \frac{\mu_0}{4\pi} I_2 I_1 \oint_{L_1} \oint_{L_2} \frac{d\vec{l}_2 \times [d\vec{l}_1 \times (\vec{r}-\vec{r}')]}{|\vec{r}-\vec{r}'|^2}. \tag{10.40}$$

Let's consider,

$$\frac{d\vec{l}_2 \times [d\vec{l}_1 \times (\vec{r}-\vec{r}')]}{|\vec{r}-\vec{r}'|^2} = -\left(d\vec{l}_2 \cdot d\vec{l}_1\right)\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} + d\vec{l}_1\left[\frac{d\vec{l}_2 \cdot (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right], \tag{10.41}$$

The loop integration of the second term in **Equation 10.41** is zero since it is an expression for a gradient. Therefore,

$$\vec{F}_2 = -\frac{\mu_0}{4\pi} I_2 I_1 \oint_{L_1} \oint_{L_2} d\vec{l}_2 \cdot d\vec{l}_1 \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}. \tag{10.42}$$

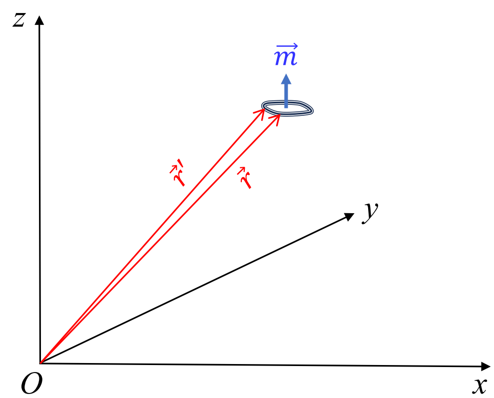

Fig. 10.5 The magnetic force on a tiny magnetic dipole.

❖ The magnetic force on a magnetic dipole

Since $\vec{F}_B = \iiint_V \vec{J}(\vec{r}') \times \vec{B}(\vec{r}')\, dV'$, for a tiny magnetic moment as shown in **Figure 10.5**, the magnetic field $\vec{B}(\vec{r}')$ at the vicinity of the magnetic moment can be expanded as,

$$\vec{B}(\vec{r}') = \vec{B}(\vec{r}) + [(\vec{r}' - \vec{r}) \cdot \nabla]\vec{B}(\vec{r}) + \cdots. \tag{10.43}$$

Therefore,

$$\begin{aligned}\vec{F}_B &= \iiint_V \vec{J}(\vec{r}') \times \{\vec{B}(\vec{r}) + [(\vec{r}' - \vec{r}) \cdot \nabla]\vec{B}(\vec{r}) + \cdots\}\, dV' \\ &= \iiint_V \vec{J}(\vec{r}') \times \vec{B}(\vec{r})\, dV' + \iiint_V \vec{J}(\vec{r}') \times [(\vec{r}' - \vec{r}) \cdot \nabla]\vec{B}(\vec{r})\, dV' + \cdots \\ &\approx \iiint_V \vec{J}(\vec{r}') \times (\vec{r}' \cdot \nabla)\vec{B}(\vec{r})\, dV' .\end{aligned} \tag{10.44}$$

Since $(\vec{r}' \cdot \nabla)\vec{B}(\vec{r}) = \nabla[\vec{r}' \cdot \vec{B}(\vec{r})] - \vec{r}' \times [\nabla \times \vec{B}(\vec{r})]$, and the second term on the right-hand side is zero. Thus,

$$\vec{F}_B = \iiint_V \vec{J}(\vec{r}') \times \nabla[\vec{r}' \cdot \vec{B}(\vec{r})]\, dV'. \tag{10.45}$$

According to the following identity $\nabla \times (C\vec{F}) = C\nabla \times \vec{F} + \nabla C \times \vec{F}$, and let $C = \vec{r}' \cdot \vec{B}(\vec{r})$, $\vec{F} = \vec{J}(\vec{r}')$,

$$\vec{J}(\vec{r}') \times \nabla[\vec{r}' \cdot \vec{B}(\vec{r})] = \vec{r}' \cdot \vec{B}(\vec{r})\nabla \times \vec{J}(\vec{r}') - \nabla \times \{[\vec{r}' \cdot \vec{B}(\vec{r})]\vec{J}(\vec{r}')\}. \tag{10.46}$$

The first term on the right-hand side of **Equation 10.46** is zero, therefore,

$$\vec{F}_B = -\iiint_V \nabla \times \{[\vec{r}' \cdot \vec{B}(\vec{r})]\vec{J}(\vec{r}')\}\, dV' = -\nabla \times \iiint_V [\vec{r}' \cdot \vec{B}(\vec{r})]\vec{J}(\vec{r}')\, dV'. \tag{10.47}$$

Let's look at the integration term. Similar to the case for multipole expansion, **Equation 10.12**,

$$\vec{C}\cdot\{\vec{B}(\vec{r})\times[\vec{J}(\vec{r}')\times\vec{r}']\} = [\vec{C}\cdot\vec{J}(\vec{r}')][\vec{B}(\vec{r})\cdot\vec{r}'] - (\vec{C}\cdot\vec{r}')[\vec{B}(\vec{r})\cdot\vec{J}(\vec{r}')], \tag{10.48}$$

$$\nabla'\cdot\{(\vec{C}\cdot\vec{r}')[\vec{B}(\vec{r})\cdot\vec{r}']\vec{J}(\vec{r}')\} = [\vec{C}\cdot\vec{J}(\vec{r}')][\vec{B}(\vec{r})\cdot\vec{r}'] + (\vec{C}\cdot\vec{r}')[\vec{B}(\vec{r})\cdot\vec{J}(\vec{r}')] + (\vec{C}\cdot\vec{r}')[\vec{B}(\vec{r})\cdot\vec{r}'][\nabla'\cdot\vec{J}(\vec{r}')]. \tag{10.49}$$

Adding **Equations 10.48** and **10.49** together, one has,

$$[\vec{C}\cdot\vec{J}(\vec{r}')][\vec{B}(\vec{r})\cdot\vec{r}'] = \frac{1}{2}\vec{C}\cdot\{\vec{B}(\vec{r})\times[\vec{J}(\vec{r}')\times\vec{r}']\} + \frac{1}{2}\nabla'\cdot\{(\vec{C}\cdot\vec{r}')[\vec{B}(\vec{r})\cdot\vec{r}']\vec{J}(\vec{r}')\}. \tag{10.50}$$

Therefore,

$$\iiint_V [\vec{r}'\cdot\vec{B}(\vec{r})]\vec{J}(\vec{r}')\,dV' = \vec{m}\times\vec{B}(\vec{r}), \tag{10.51}$$

and,

$$\vec{F}_B = -\nabla\times[\vec{m}\times\vec{B}(\vec{r})] = (\vec{m}\cdot\nabla)\vec{B}(\vec{r}) - \vec{m}[\nabla\cdot\vec{B}(\vec{r})] = \nabla[\vec{m}\cdot\vec{B}(\vec{r})]. \tag{10.52}$$

Equation 10.52 has a very similar form comparing to the electrostatic force for an electric dipole, $\vec{F}_E = \nabla[\vec{p}\cdot\vec{E}(\vec{r})]$.

❖ The force between two magnetic dipoles

For two magnetic dipoles, one will generate a magnetic field at the location of the other dipole as shown in **Figure 10.6**, therefore there will be a magnetic force acting on each other. The magnetic field generated by the first magnetic dipole $\vec{m}_1$ is,

$$\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{r}(\hat{r}\cdot\vec{m}_1)-\vec{m}_1}{r^3}, \tag{10.53}$$

And

$$\vec{F}_2 = (\vec{m}_2\cdot\nabla)\vec{B}_1(\vec{r}) = \frac{3\mu_0}{4\pi r^4}[(\vec{m}_1\cdot\vec{m}_2)\hat{r} + (\vec{m}_1\cdot\hat{r})\vec{m}_2 + (\vec{m}_2\cdot\hat{r})\vec{m}_1 - 5(\vec{m}_1\cdot\hat{r})(\vec{m}_2\cdot\hat{r})\hat{r}]. \tag{10.54}$$

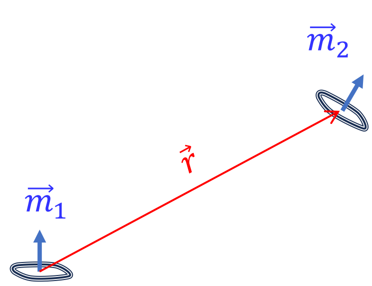

Fig. 10.6 The magnetic force between two magnetic dipoles.

10.3 Magnetic Energy

For magnetism, since there is no magnetic monopole, the fundamental interaction for a magnetic system is between a magnetic field and a magnetic dipole, which is described by **Equation 10.52**. Since $\vec{F}_B = \nabla\left[\vec{m}\cdot\vec{B}(\vec{r})\right]$, according to the force-potential energy relationship, i.e., $\vec{F} = -\nabla U(\vec{r})$, the magnetic potential energy can be defined as,

$$U_B(\vec{r}) = -\vec{m}\cdot\vec{B}(\vec{r}). \qquad (10.55)$$

Such a definition has the same form as for the electric potential energy of an electric dipole, $U_E(\vec{r}) = -\vec{p}\cdot\vec{E}(\vec{r})$. Therefore, the magnetic dipole-dipole interaction energy (refer to Figure 10.6) can be written as

$$U_B(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{\vec{m}_1\cdot\vec{m}_2 - 3(\hat{r}\cdot\vec{m}_1)(\hat{r}\cdot\vec{m}_2)}{r^3} - \frac{8\pi}{3}(\vec{m}_1\cdot\vec{m}_2)\delta(\vec{r})\right]. \qquad (10.56)$$

For a collective *N* magnetic dipole distribution, the total magnetic interaction energy of the system can be written as

$$U_B(\vec{r}) = -\frac{1}{2}\sum_{i=1}^{N}\vec{m}_i\cdot\vec{B}(\vec{r}_i). \qquad (10.57)$$