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Chapter 7
Boundary Value Problem

7.1 General description

The fundamental challenge in electrostatics is to determine the electrostatic potential for any arbitrary charge distribution across different material systems with varying geometries. Ideally, once the charge distribution $\rho(\vec{r}')$ of an object is known, the potential $\varphi(\vec{r})$ can be determined using the integral

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_{V'} \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' . \qquad (3.13)$$

However, if the charge distribution is unknown or if there are interactions among objects causing charge redistribution through induced charges, **Equation 3.13** is not applicable. In such cases, alternative methods are required. Fortunately, Gauss's law provides that the electrostatic potential $\varphi(\vec{r})$ must satisfy Poisson's equation (**Equation 3.15**),

$$\nabla^2\varphi(\vec{r}) = -\frac{\rho_f(\vec{r})}{\varepsilon\varepsilon_0}. \qquad (3.15)$$

At boundaries between two different materials, the potential must satisfy the following boundary conditions (BCs) (**Equation 3.25**),

$$\begin{cases} \varepsilon_2\dfrac{\partial\varphi_2}{\partial n_2} + \varepsilon_1\dfrac{\partial\varphi_1}{\partial n_1} = \dfrac{\sigma_f}{\varepsilon_0} . \\ \varphi_2(\vec{r}_s) = \varphi_1(\vec{r}_s) \end{cases} \qquad (3.25)$$

If $\rho_f = 0$ in the space, the Poisson's equation is reduced to Laplace's equation

$$\nabla^2\varphi(\vec{r}) = 0. \qquad (3.16)$$

To determine the potential, we can solve the Laplace's equation, whose solutions are defined by specific boundary conditions and initial values. Given that we are dealing with a time-independent partial differential equation (PDE), BCs are especially crucial for finding a unique solution.

There are five types of boundary conditions commonly used in solving Laplace's equation:

1) <u>*Dirichlet BC*</u>: The potential $\varphi(\vec{r})$ is specified on the boundary, so

$$\varphi(\vec{r})|_{\vec{r}_1} = f(\vec{r}). \tag{7.1}$$

2) *Neumann BC*: The derivative of the potential normal to the boundary is specified, expressed as

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\vec{r}_1} = f(\vec{r}). \tag{7.2}$$

3) *Robin BC*: A linear combination of the potential and its normal derivative is specified, defined by

$$\left.\left[C_0\varphi(\vec{r}) + C_1\frac{\partial\varphi}{\partial n}\right]\right|_{\vec{r}_1} = f(\vec{r}). \tag{7.3}$$

4) *Mixed BCs*: Both the potential and a combination of the potential and its normal derivative are specified on different parts of the boundary

$$\varphi(\vec{r})|_{\vec{r}_1} = f(\vec{r}) \text{ and } \left.\left[C_0\varphi(\vec{r}) + C_1\frac{\partial\varphi}{\partial n}\right]\right|_{\vec{r}_1} = g(\vec{r}). \tag{7.4}$$

5) *Cauchy BCs*: Both the potential and its normal derivative are specified simultaneously on the same boundary, given by

$$\varphi(\vec{r})|_{\vec{r}_1} = f(\vec{r}) \text{ and } \left.\frac{\partial\varphi}{\partial n}\right|_{\vec{r}_1} = g(\vec{r}). \tag{7.5}$$

Here are examples of electrostatic systems that satisfy each of the five types of boundary conditions (BCs):

1) *Dirichlet BC*: A grounded, conducting sphere in an external electric field. Suppose a conducting sphere is placed in a uniform electric field. The surface of the conductor is grounded, meaning the potential $\varphi(\vec{r})$ is zero on the sphere's surface. This condition can be represented as $\varphi(\vec{r})|_{\vec{r}_1} = 0$, where $\vec{r}_1$ is any point on the sphere's surface. Here, the Dirichlet BC fixes the potential to a known function (in this case, zero) on the conductor's surface.

2) *Neumann BC*: An isolated, charged conducting plane. Consider a large conducting plane with a uniform surface charge density. Since this surface charge density directly relates to the electric field perpendicular to the plane, we can express this as $\left.\frac{\partial\varphi}{\partial n}\right|_{\vec{r}_1} = \frac{\sigma}{\varepsilon_0}$ where σ is the surface charge density and $\vec{r}_1$ is any point on the plane. The Neumann BC specifies the normal derivative of the potential, which is equivalent to the electric field near the surface of the conductor.

3) *Robin BC*: An imperfectly conducting (dielectric) sphere in a uniform external electric field. When a dielectric sphere is placed in an electric field, the potential on its surface can be expressed as a weighted combination of the potential and its normal derivative due to partial reflection of the field at the boundary. This is often modeled as $C_0\varphi(\vec{r}) + C_1\frac{\partial\varphi}{\partial n} = f(\vec{r})$ at $\vec{r}_1$, where the constants C_0 and C_1 depend on the dielectric properties of the sphere. The Robin boundary condition reflects a semi-

conducting or dielectric behavior, where both the potential and the flux influence the behavior on the surface.

4) <u>*Mixed BCs*</u>: A grounded conducting plane with a nearby charged dielectric slab. Imagine a grounded conducting plane with a dielectric slab placed near it. On one part of the boundary, the conductor imposes a fixed potential (Dirichlet condition), while on another boundary near the dielectric, the potential obeys a Robin condition due to the dielectric's partial influence on the field. This could be expressed as $\varphi(\vec{r})|_{\vec{r}_1} = 0$ on the conductor's surface (Dirichlet) and $C_0\varphi(\vec{r}) + C_1\frac{\partial\varphi}{\partial n} = g(\vec{r})$ at the dielectric interface (Robin).

5) *Cauchy BCs*: A system involving both specified surface potential and surface electric field, such as a grounded conductor with a known electric field just outside the surface. In this scenario, both the potential φ and its normal derivative $\frac{\partial\varphi}{\partial n}$ (related to the electric field) are specified at the boundary. An example could be a region around a conducting surface where we know both the surface potential (say, grounded at $\varphi = 0$) and the electric field, perhaps due to nearby charges or imposed fields. This is relevant when modeling regions with highly controlled potentials and fields, such as those in certain electrostatic applications or specific laboratory setups.

In most practical cases, Dirichlet and Neumann boundary conditions are the primary types encountered. These boundary conditions are essential in constraining the solution space, making it possible to solve for the potential accurately within the system.

From mathematical physics, under each of the five types of boundary conditions, the following key properties of solutions can be proven:

1) Uniqueness of solutions

Under any of these boundary conditions, the electrostatic potential solution is unique. This means that if a solution satisfies the boundary conditions and the governing equations (such as Laplace's or Poisson's equation), then it is the only possible solution for the given physical setup. Uniqueness ensures that the solution corresponds reliably to the physical configuration and is fundamental in fields like electrostatics, where potential ambiguity can lead to non-physical results.

2) Completeness of solutions

Completeness implies that the set of solutions obtained under the boundary conditions spans the entire solution space. In practical terms, this means any potential distribution satisfying the boundary conditions can be represented by a combination of solutions (such as eigenfunctions) derived under these conditions. Completeness ensures that the solutions are sufficiently general to describe all physically possible configurations for a given problem.

3) Orthogonality of eigenstates

For boundary-value problems, solutions (often termed "eigenstates" or "eigenfunctions") under the specified conditions exhibit orthogonality. This orthogonality property is essential when decomposing complex potential distributions into simpler components. In electrostatics, orthogonal eigenfunctions help simplify solutions in complex geometries, making it possible to represent the potential as a series expansion of simpler functions that individually satisfy the boundary conditions.

These properties of uniqueness, completeness, and orthogonality underpin the mathematical foundation of electrostatic solutions, allowing for systematic and predictable approaches to finding solutions that accurately reflect real-world physical systems.

7.2 Boundary Value Problems in Rectangular Geometry

A common method for solving PDE is the separation of variables. For Laplace's equation with rectangular BCs, we can work in Cartesian coordinates to express **Equation 3.16** as,

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0. \tag{7.6}$$

Assuming a solution of the form $\varphi(x,y,z)=X(x)Y(y)Z(z)$, **Equation 7.6** can be rewritten as,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0. \tag{7.7}$$

Since each term in **Equation 7.7** depends on a single variable independently, we can separate the equation as,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=\alpha^2,\qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=\beta^2,\qquad \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=\gamma^2, \tag{7.8}$$

with the condition that

$$\alpha^2+\beta^2+\gamma^2=0. \tag{7.9}$$

Here α^2, β^2, and γ^2 can take positive or negative values, which means that $\alpha,\beta,$ and γ could be complex numbers. This approach enables us to reduce the original PDE into a set of ordinary differential equations (ODEs), each depending on only one of the spatial variables, which can be solved separately under the specified boundary conditions.

The general solutions for the functions $X(x), Y(y),$ and $Z(z)$ can be written as,

$$X_\alpha(x)=\begin{cases} A_0+B_0x, & \alpha=0\\ A_\alpha e^{\alpha x}+B_\alpha e^{-\alpha x}, & \alpha\neq 0\end{cases}, \tag{7.10}$$

$$Y_\beta(y)=\begin{cases} C_0+D_0y, & \beta=0\\ C_\beta e^{\beta y}+D_\beta e^{-\beta y}, & \beta\neq 0\end{cases}, \tag{7.11}$$

$$Z_\gamma(z) = \begin{cases} E_0 + F_0 z, & \gamma = 0 \\ E_\gamma e^{\gamma z} + F_\gamma e^{-\gamma z}, & \gamma \neq 0 \end{cases}, \quad (7.11)$$

These solutions correspond to different cases for α, β, and γ, depending on whether the separation constants are zero or non-zero. When the separation constant is zero, the solution is linear, while for non-zero values, the solution can be expressed as exponential functions.

The final solution for potential $\varphi(x, y, z)$ is then given as a linear combination of all possible products of these solutions $X(x)$, $Y(y)$, and $Z(z)$,

$$\varphi(x, y, z) = \sum_\alpha \sum_\beta \sum_\gamma X_\alpha(x) Y_\beta(y) Z_\gamma(z) \delta(\alpha^2 + \beta^2 + \gamma^2), \quad (7.12)$$

where $\delta(\alpha^2 + \beta^2 + \gamma^2)$ serves as a selection function,

$$\delta(\alpha^2 + \beta^2 + \gamma^2) = \begin{cases} 1, & \text{if } \alpha^2 + \beta^2 + \gamma^2 = 0 \\ 0, & \text{if } \alpha^2 + \beta^2 + \gamma^2 \neq 0 \end{cases}.$$

This ensures that only combinations of α, β, and γ satisfying $\alpha^2 + \beta^2 + \gamma^2 = 0$; contribute to the final solution. The coefficients A_α, B_α, C_β, D_β, E_γ, and F_γ are determined by applying the specific BCs for the problem, which provide the necessary constraints to fully define the potential in the region of interest.

Example 7.1 *Find the electrostatic potential $\varphi(x, y, z)$ inside a rectangular potential box as shown in **Fig. 7.1**, where the potential is zero on five surfaces and is given by $\varphi = V(x, y)$ at the top surface.*

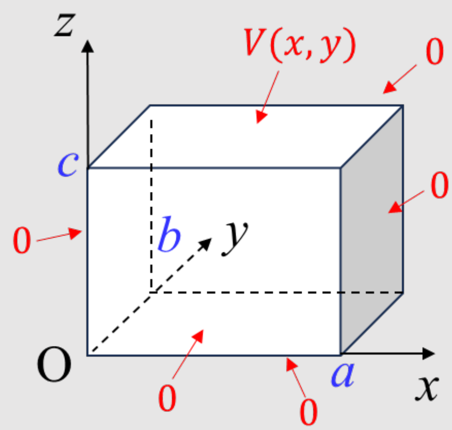


Fig. 7.1 A rectangular potential box.

Discussion: The geometry of the system satisfies the conditions needed for applying the separation of variables in Cartesian coordinate, as outlined in **Equation 7.7**. This allows us to express the general solution using **Equation 7.12**. To determine the specific solution, we must mathematically define the BCs to find all the coefficients A_α, B_α, C_β, D_β, E_γ, and F_γ.

Solution: The BCs for this problem can be specified as,

$$\begin{cases} \varphi(0,y,z)=0 \text{ and } \varphi(a,y,z)=0 \\ \varphi(x,0,z)=0 \text{ and } \varphi(x,b,z)=0 \\ \varphi(x,y,0)=0 \text{ and } \varphi(x,y,c)=V(x,y) \end{cases}$$

Based on these BCs and using the general form of the solution given by **Equations 7.10-7.11**, we can establish the conditions,

$$\begin{cases} X(0)=0 \\ X(a)=0 \end{cases}, \quad \begin{cases} Y(0)=0 \\ Y(b)=0 \end{cases}, \quad Z(0)=0$$

We must ensure that $\alpha \neq 0$, $\beta \neq 0$, and $\gamma \neq 0$; otherwise, the solution would reduce to the trivial case $\varphi(x,y,z) = \text{constant}$.

Step 1: Solving for $X(x)$

For $X(x)$, using **Equation 7.10**, we obtain

$$\begin{cases} A_\alpha + B_\alpha = 0, & \text{for } x=0 \\ A_\alpha e^{\alpha a} + B_\alpha e^{-\alpha a} = 0, & \text{for } x=a \end{cases}$$

From the first equation, we get

$$A_\alpha = -B_\alpha$$

Substituting this into the second equation give,

$$A_\alpha\left(e^{\alpha a} - e^{-\alpha a}\right) = 0$$

Since A_α cannot be zero (otherwise, $X(x) = 0$, leading to $\varphi(x,y,z) = 0$), we require,

$$e^{\alpha a} - e^{-\alpha a} = 0$$

This implies that α must be purely imaginary, so we write,

$$\alpha = i\alpha'$$

Thus, we have,

$$\sin(\alpha' a) = 0$$

Hence, the allowed values of α' are,

$$\alpha' = \frac{n\pi}{a}, \qquad n = 1,2,3,\cdots$$

The solution for $X(x)$ can then be written as,

$$X_n(x) = A_n \sin\left(\frac{n\pi}{a}x\right)$$

Step 2: Solving for $Y(y)$

Following a similar approach for $Y(y)$, we find

$$Y_m(y) = C_m \sin\left(\frac{m\pi}{b}y\right), \qquad m = 1,2,3,\cdots$$

Thus, we can write

$$\beta = i\beta' = \frac{m\pi}{b}$$

Step 3: Solving for $Z(z)$

Using **Equation 7.9**, we get

$$\gamma^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 = 0$$

Therefore,

$$\gamma_{nm} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

Since $Z(0) = 0$, we obtain

$$E_\gamma = -F_\gamma$$

Thus, the general solution for $Z(z)$ becomes

$$Z(z) = E_\gamma \sinh(\gamma z)$$

Step 4: Combining the solutions

Combining the solutions for $X(x), Y(y),$ and $Z(z)$, the potential can be expressed as

$$\varphi(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh(\gamma_{nm}z)$$

Here, the coefficient A_{nm} encompass all factors, such that $A_{nm} = A_n C_m E_\gamma$.

Step 5: Applying the final BC

To determine the coefficient A_{nm}, we apply the BC $\varphi(x, y, c) = V(x, y)$, yielding

$$V(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh(\gamma_{nm}c)$$

Step 6: Using orthogonality to solve for A_{nm}

The orthogonality of of sine functions allows us to isolate A_{nm},

$$\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n'\pi}{a}x\right)dx = \frac{a}{2}\delta_{nn'}$$

$$\int_0^b \sin\left(\frac{m\pi}{b}y\right)\sin\left(\frac{m'\pi}{b}y\right)dy = \frac{b}{2}\delta_{mm'}$$

We have

$$\int_0^a dx \int_0^b dy\, V(x,y) \sin\left(\frac{n'\pi}{a}x\right) \sin\left(\frac{m'\pi}{b}y\right) =$$

$$\int_0^a dx \int_0^b dy \sin\left(\frac{n'\pi}{a}x\right) \sin\left(\frac{m'\pi}{b}y\right) \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh(\gamma_{nm}c) =$$

$$A_{n'm'} \sinh(\gamma_{n'm'}c) \frac{ab}{4}$$

Thus, the coefficient A_{nm} can be found as

$$A_{nm} = \frac{4}{ab \sinh(\gamma_{nm}c)} \int_0^a dx \int_0^b dy\, V(x,y) \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right)$$

This solution provides a complete expression for the electrostatic potential inside the box, satisfying the specified boundary conditions.

7.3 Boundary Value Problems in Geometry of Azimuthal Symmetry

When the potential of an electrostatic system exhibits azimuthal symmetry, spherical coordinates are a natural choice for simplifying the problem. In spherical coordinates as shown in **Figure 7.2**, the potential is expressed as $\varphi(x,y,z) = \varphi(r,\theta,\phi)$, where r is the radial distance, θ is the polar angle, and ϕ is the azimuthal angle. For a system with azimuthal symmetry, the potential is independent of ϕ, so$\varphi(r,\theta,\phi) = \varphi(r,\theta)$.

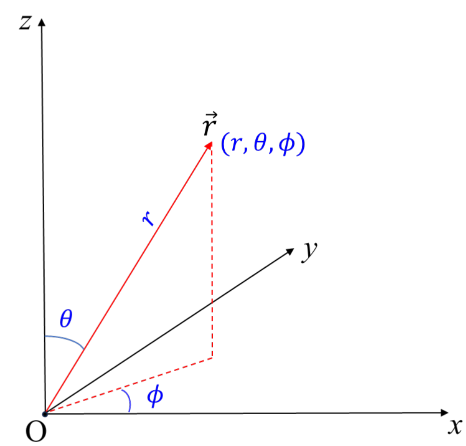


Fig. 7.2 Spherical coordinates.

In spherical coordinates with the azimuthal symmetry, the Laplace's equation can be written as,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \varphi}{\partial \theta}\right) = 0. \tag{7.13}$$

To solve this equation, we use the method of separation of variables by letting $\varphi(r,\theta) = R(r)\Theta(\theta)$, leading to the following separate ODEs

$$\begin{cases} \dfrac{d}{dr}\left(r^2 \dfrac{dR}{dr}\right) - \kappa R = 0 \\ \dfrac{d}{dx}\left[(1-x^2)\dfrac{d\Theta}{dx}\right] + \kappa\Theta = 0 \end{cases}, \tag{7.14}$$

where $x = \cos\theta$, and κ is a separation constant. The general solutions for **Equation 7.14** can be written as,

$$R_l(r) = A_l r^l + B_l r^{-(l+1)}, \tag{7.15}$$

$$\Theta_l(\theta) = C_l P_l(\cos\theta) + D_l Q_l(\cos\theta), \tag{7.16}$$

where $\kappa = l(l+1)$, $P_l(\cos\theta)$ and $Q_l(\cos\theta)$ are Legendre functions of the fist and second kinds. The general solution for $\varphi(r,\theta)$ can be expressed as,

$$\varphi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]\left[C_l P_l(\cos\theta) + D_l Q_l(\cos\theta)\right]. \tag{7.17}$$

Here, A_l, B_l, C_l, D_l, are coefficients that must be determined by applying the BCs specific to the physical problem at hand.

Section 4.2.2 covers some fundamental properties of the Legendre polynomials $P_l(x)$. The Legendre functions of the second kind, $Q_l(x)$, are solutions to Legendre's differential equation, like the Legendre polynomials $P_l(x)$. However, as shown below, $Q_l(x)$ exhibit singularities at $x = \pm 1$, making them unsuitable for representing the electrostatic potential in most physical situations where the potential must remain finite on the spherical boundary. Consequently, $Q_l(x)$ is typically not included in the general solution for the electrostatic potential.

Legendre Function of the Second Kind

The Legendre functions of the second kind $Q_l(x)$ can be represented as the following when l is an integer,

$$Q_l(x) = \frac{1}{2}P_l(x)\ln\left(\frac{1+x}{1-x}\right) - \sum_{k=1}^{l}\frac{1}{k}P_{l-k}(x)P_{k-1}(x).$$

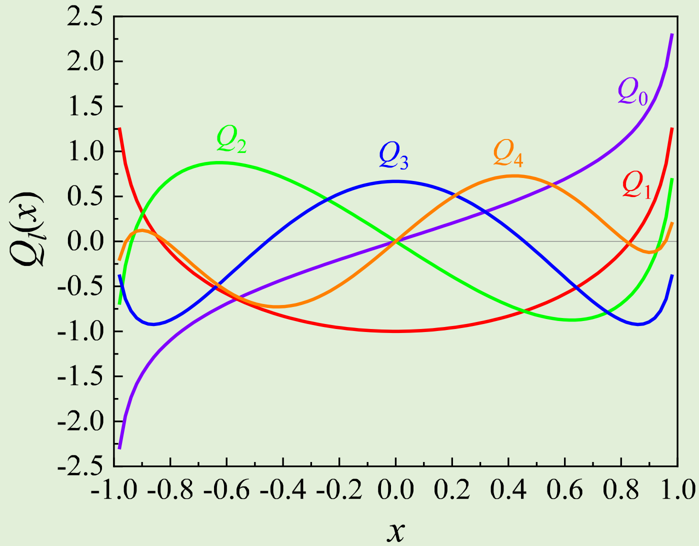
Typically, $Q_l(x)$ exhibits singularities at $x = \pm 1$. For the firs 5 $Q_l(x)$, they are

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

$$Q_2(x) = \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}$$

$$Q_3(x) = \frac{5x^3 - 3x}{6}\ln\left(\frac{1+x}{1-x}\right) - \frac{5x^2}{2} + \frac{2}{3}$$
$$Q_4(x) = \frac{35x^4 - 30x^2 + 3}{24}\ln\left(\frac{1+x}{1-x}\right) - \frac{35x^3}{12} + \frac{25x}{8}$$



The function $Q_l(x)$ diverges at $x = \pm 1$, so it is undefined at these points. This divergence takes a logarithmic form. In applications where real values are required, $Q_l(x)$ is often used only in the domain $x > 1$.

In problems with azimuthal symmetry, the electrostatic potential $\varphi(r, \theta)$ usually takes the following form,

$$\varphi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta). \tag{7.18}$$

The coefficients A_l and B_l can be obtained via the orthogonality property of Legendre polynomials under specific BCs,

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \frac{2}{2l+1}\delta_{ll'}, \tag{7.19}$$

or

$$\int_{0}^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta = \frac{2}{2l+1}\delta_{ll'}. \tag{7.20}$$

Example 7.2 *Find the electrostatic potential and electric field of a conducting sphere (with a radius of a) placed in a uniform electric field, shown in* ***Fig. 7.3****.*

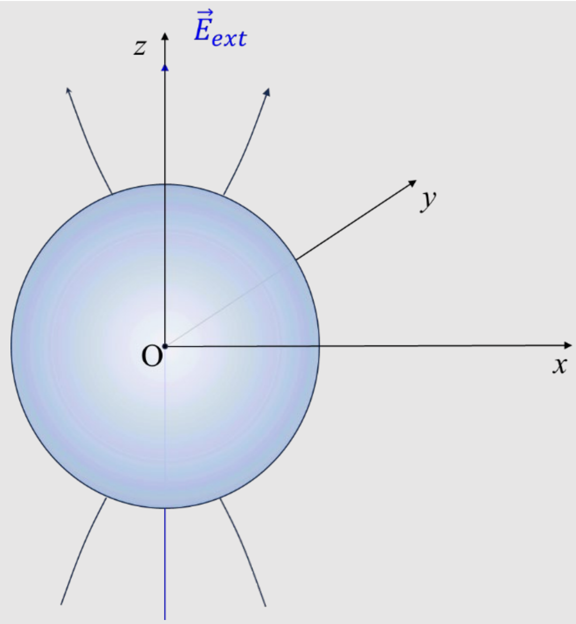


Fig. 7.3 A conducting sphere in a uniform electric field.

Discussion: This problem involves a conducting sphere placed in a uniform electric field, similar to the scenario discussed in **Example 5.1**. The system exhibits azimuthal symmetry, meaning that the potential $\varphi(r,\theta)$ is independent of the azimuthal angle ϕ. Inside the conducting sphere, the potential remains constant, since a conductor in electrostatic equilibrium is an equipotential, and the electric field is zero. The external uniform electric field E_0 induces a redistribution of charges on the sphere's surface, effectively creating an induced dipole. The resulting potential outside the sphere behaves like that of a dipole, decaying as $1/r^2$. Far from the sphere ($r \to \infty$), the potential does not approach zero, but rather converges to $\varphi(r \to \infty, \theta) = -E_0 z = -E_0 r \cos\theta$.

Solution: Outside the conducting sphere, the potential satisfies Laplace's equation. The boundary conditions for this problem can be stated as,

$$\varphi(r \to \infty, \theta) = -E_0 z = -E_0 r \cos\theta = -\,E_0 r P_1(\cos\theta)$$

$$\varphi(a,\theta) = 0$$

Here $\varphi(a,\theta) = 0$ establishes the conducting sphere as the reference potential.

Step 1: General solution for the potential

The general solution for Laplace's equation is given by **Equation 7.18**. Applying the BC at $r \to \infty$, we have

$$\varphi(r \to \infty, \theta) = \sum_{l=0}^{\infty} A_l r^l \, P_l(\cos\theta)$$

Thus,

$$A_0 + A_1 r P_1(\cos\theta) + \sum_{l=2}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r P_1(\cos\theta)$$

Due to the orthogonality of $P_1(\cos\theta)$, matching the coefficients in front of each $P_l(\cos\theta)$ on both side of above equation, we find

$$A_1 = -E_0, A_0 = 0, A_l = 0 \text{ for } l \geq 2$$

Therefore, the potential can be written as,

$$\varphi(r,\theta) = -E_0 r P_1(\cos\theta) + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

Step 2: Applying the BC at $r = a$

At the surface of the sphere, $\varphi(a,\theta) = 0$, giving

$$-E_0 a P_1(\cos\theta) + \sum_{l=0}^{\infty} B_l a^{-(l+1)} P_l(\cos\theta) = 0$$

Again, using the orthogonality property of Legendre polynomials, we find that only B_1 is non-zero. Therefore, we have

$$-E_0 a P_1(\cos\theta) + B_1 a^{-2} P_1(\cos\theta) = 0$$

Solving for B_1, we obtain

$$B_1 = E_0 a^3$$

Step 3: Final expression for the potential

The final solution for the potential outside the sphere is given by

$$\varphi(r,\theta) = -E_0 r \cos\theta + \frac{E_0 a^3}{r^2} \cos\theta$$

The second term, $\frac{E_0 a^3}{r^2} \cos\theta$, represents the induced dipole potential.

Step 4: Induced dipole moment and polarizability

The induced potential term can be interpreted as a dipole potential,

$$\frac{E_0 a^3}{r^2} \cos\theta = \frac{p_{induced}}{4\pi\varepsilon_0 r^2} \cos\theta$$

Thus, the induced dipole moment is

$$p_{induced} = 4\pi\varepsilon_0 a^3 E_0$$

The polarizability of the conducting sphere is given by,

$$\alpha = 4\pi a^3$$

This result matches the value derived in **Example 5.1**.

Step 5: Electric field calculation

The electric field outside the sphere can be found by taking the negative gradient of the potential and has two components,

$$\begin{cases} E_r(r,\theta) = -\frac{\partial\varphi}{\partial r} = E_0\left(1+\frac{2a^3}{r^3}\right)\cos\theta \\ E_\theta(r,\theta) = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = -E_0\left(1-\frac{a^3}{r^3}\right)\sin\theta \end{cases}$$

Step 6: Induced surface charge density

The surface charge density on the sphere is related to the discontinuity in the normal electric field at the surface,

$$\sigma(\theta) = -\varepsilon_0 \left.\frac{\partial\varphi(r,\theta)}{\partial r}\right|_{r=a} = 3\varepsilon_0 E_0 \cos\theta$$

This result indicates that the surface charge density varies with $\cos\theta$, concentrating more charge at the poles of the sphere.

Example 7.3 *A point dipole $\vec{p}_0$, oriented along the +z -axis, is placed at the center of a homogeneous dielectric sphere of radius a and dielectric constant ε_1. The sphere is immersed in an external medium with dielectric constant ε_2, as shown in* ***Fig. 7.4***. *The goal is to find the electrostatic potential both inside and outside the sphere, as well as the polarization charge density on the sphere's surface.*

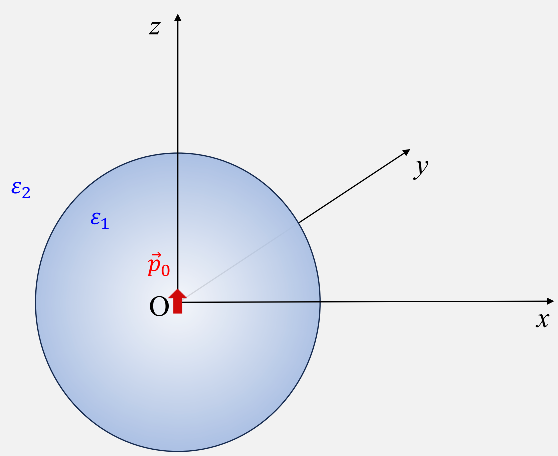


Fig. 7.4 A dielectric sphere containing a point dipole, immersed in a medium with a different dielectric constant.

Discussion: The presence of a point dipole at the center implies that the potential cannot be determined using Laplace's equation everywhere, since Laplace's equation applies only in regions without free charges. However, apart from the origin, there are no free charges inside or outside the dielectric sphere, making

Laplace's equation valid in these regions. The solution can be approached using the principle of superposition. The total potential inside the sphere, φ_{in}, can be viewed as a combination of the potential generated by the point dipole φ_d and the potential φ'_{in} induced by the polarization of the sphere, including any surface polarization charges. Outside the sphere, the potential φ_{out} results solely from the polarization of the sphere in the external dielectric. Thus, both φ'_{in} and φ_{out} satisfy Laplace's equation, while the potential φ_d is described by the dipole field, **Equation 4.29**.

Solution: The problem exhibits azimuthal symmetry due to the alignment of the dipole along the *z*-axis, allowing us to express the potentials in terms of spherical coordinates (r, θ).

Step 1: Potential inside the sphere

The potential inside the sphere is the sum of the potential due to the point dipole and the induced potential φ'_{in}

$$\varphi_{in} = \frac{\vec{p}_0 \cdot \vec{r}}{4\pi\varepsilon_1 r^3} + \varphi'_{in} \quad \text{for } r \le a$$

The dipole potential is given by

$$\varphi_d = \frac{\vec{p}_0 \cdot \vec{r}}{4\pi\varepsilon_1 r^3}$$

Step 2: Boundary conditions

To solve for the potentials, we use the following BCs:

(1) *At infinity*: The potential outside the sphere vanishes as $r \to \infty$, *i.e.*, when $r \to \infty$, $\varphi_{out} \to 0$
(2) *At the center*: The induced potential φ'_{in} must be finite at $r = 0$
(3) *Continuity of potential at* $r = a$: The potential is continuous across the boundary, $\varphi_{in}(a, \theta) = \varphi_{out}(a, \theta)$
(4) *Continuity of the normal component of displacement field*: The normal component of the electric displacement field is continuous across the boundary: $\varepsilon_2 \frac{\partial \varphi_{out}}{\partial r}\Big|_{r=a} = \varepsilon_1 \frac{\partial \varphi_{in}}{\partial r}\Big|_{r=a}$

Step 3: General solutions for φ'_{in} and φ_{out}

Since both φ'_{in} and φ_{out} satisfy the Laplace's equation, according to **Equation 7.17**, their general solutions are,

$$\begin{cases} \varphi'_{in} = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta) & \text{for } r \le a \\ \varphi_{out} = \sum_{l=0}^{\infty} \left[C_l r^l + D_l r^{-(l+1)}\right] P_l(\cos\theta) & \text{for } r > a \end{cases}$$

Step 4: Applying boundary conditions

Since the potential must vanish at infinity, i.e., implementing the BC#1, we can obtain $C_l = 0$ for all l.

To ensure that φ'_{in} remains finite at the origin, i.e., the BC#2, we have $B_l = 0$ for all l.

Thus, the potentials reduce to,

$$\begin{cases} \varphi'_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & \text{for } r \le a \\ \varphi_{out} = \sum_{l=0}^{\infty} D_l r^{-(l+1)} P_l(\cos\theta) & \text{for } r > a \end{cases}$$

BC#3 and BC#4 give two equations,

$$\begin{cases} \dfrac{p_0 \cos\theta}{4\pi\varepsilon_1 a^2} + \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = \sum_{l=0}^{\infty} D_l a^{-(l+1)} P_l(\cos\theta) \\ -\dfrac{p_0 \cos\theta}{2\pi a^3} + \sum_{l=1}^{\infty} \varepsilon_1 l A_l a^{l-1} P_l(\cos\theta) = -\sum_{l=0}^{\infty} \varepsilon_2 (l+1) D_l a^{-(l+2)} P_l(\cos\theta) \end{cases}$$

Step 5: Solving for coefficients
Comparing coefficients of $P_l(\cos\theta)$ for each $\boldsymbol{l}$, we find,

$$A_1 = \frac{p_0}{2\pi a^3} \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1(2\varepsilon_2 + \varepsilon_1)}$$

$$D_1 = \frac{3p_0}{4\pi(2\varepsilon_2 + \varepsilon_1)}$$

$$A_l = D_l = 0 \quad \text{for } l \ne 1$$

Step 6: Final solutions for the potentials
The electrostatic potentials inside and outside the sphere are given by,

$$\begin{cases} \varphi_{in} = \dfrac{p_0 \cos\theta}{4\pi\varepsilon_1 r^2} + \dfrac{\varepsilon_1 - \varepsilon_2}{2\pi(2\varepsilon_2 + \varepsilon_1)} \dfrac{p_0 \cos\theta}{r^2} & \text{for } r \le a \\ \varphi_{out} = \dfrac{3p_0 \cos\theta}{4\pi(2\varepsilon_2 + \varepsilon_1) r^2} & \text{for } r > a \end{cases}$$

Step 7: Polarization surface charge density

The surface polarization charge density at $r = a$ can be calculated using the discontinuity in the normal component of the electric displacement field

$$\sigma_P = -\hat{n} \cdot \left(\vec{P}_2 - \vec{P}_1\right) = (\varepsilon_2 - \varepsilon_0) \left.\frac{\partial \varphi_{out}}{\partial r}\right|_{r=a} - (\varepsilon_1 - \varepsilon_0) \left.\frac{\partial \varphi_{in}}{\partial r}\right|_{r=a}$$

$$= \frac{3\varepsilon_0(\varepsilon_1 - \varepsilon_2)p_0 \cos\theta}{4\pi\varepsilon_1(2\varepsilon_2 + \varepsilon_1)a^3}$$

7.4 Boundary Value Problems with Spherical Symmetry

When the potential $\varphi(r,\theta,\phi)$ in an electrostatic system exhibits spherical symmetry, solving the problem in spherical coordinates becomes the natural choice. In this coordinate system, Laplace's equation is expressed as,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} = 0. \tag{7.21}$$

To solve Laplace's equation, we use the method of separation of variables, where the potential is assumed to take the form $\varphi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$. Substituting this into Laplace's equation and separating variables gives us three ODEs

$$\begin{cases} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\lambda}{r^2}R = 0 \\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta = 0 \\ \frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \end{cases}, \tag{7.21}$$

where λ and m are separation constants. Let $\lambda = l(l+1)$, where l is a non-negative integer, leading to the following general solutions.

$$R_l(r) = A_l r^l + B_l r^{-(l+1)}, \tag{7.22}$$

$$\Phi_m(\phi) = e^{im\phi} \text{ or } e^{-im\phi} \tag{7.23}$$

$$\Theta_{lm}(\theta) = \sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta), \qquad -l \le m \le l \tag{7.24}$$

where $P_l^m(x)$ is the associated Legendre function, defined by

$$P_l^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l(x). \tag{7.25}$$

The $P_l^m(x)$ satisfies the following orthogonality conditions,

$$\int_{-1}^{1} P_l^m(x)\,P_{l'}^m(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}, \tag{7.26}$$

or

$$\int_0^\pi P_l^m(\cos\theta)\,P_{l'}^m(\cos\theta)\sin\theta\,d\theta = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}. \tag{7.26}$$

Similarly, the azimuthal solutions $\Phi_m(\phi)$ satisfy,

$$\int_0^{2\pi} e^{im\phi}e^{-im'\phi}\,d\phi = 2\pi\delta_{mm'}. \tag{7.27}$$

Combining the angular and azimuthal solutions, $\Theta_{lm}(\theta)$ and $\Phi_m(\phi)$, we define the spherical harmonics, $\mathrm{Y}_{lm}(\theta,\phi)$, as

$$\mathrm{Y}_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}. \tag{7.28}$$

The spherical harmonics form a complete set of orthogonal functions on the sphere and are widely used in solving problems with spherical symmetry. Therefore, the general solution for Laplace's equation in spherical coordinates can be written as

$$\varphi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}r^{-(l+1)}\right]\mathrm{Y}_{lm}(\theta,\phi). \tag{7.29}$$

In typical boundary value problems, the potential is expressed differently for regions inside and outside a spherical boundary,

$$\varphi(r,\theta,\phi) = \begin{cases} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}r^l\,\mathrm{Y}_{lm}(\theta,\phi), & \text{inside the sphere} \\ \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm}r^{-(l+1)}\,\mathrm{Y}_{lm}(\theta,\phi). & \text{outside the sphere} \end{cases} \tag{7.30}$$

Example 7.4 *Find the electrostatic potential $\varphi(r,\theta,\phi)$ outside a spherical shell of radius R with a given surface charge density $\sigma(\theta,\phi)$.*

Discussion: Given the spherical geometry of the problem, we can use a solution expressed in spherical harmonics (**Equation 7.29**), as they are naturally suited to this symmetry. The boundary conditions are as follows: 1) At infinity: as $r\to\infty$, $\varphi(r,\theta,\phi)$ must approach zero; 2) On the shell surface, $\sigma(\theta,\phi) = -\varepsilon_0\frac{\partial\varphi}{\partial r}\Big|_{r=R}$.

Solution: According to the BC at $r\to\infty$, $\varphi(r,\theta,\phi)\to 0$, the solution for $\varphi(r,\theta,\phi)$ can be written as (for $r>R$),

$$\varphi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm}r^{-(l+1)}\,\mathrm{Y}_{lm}(\theta,\phi)$$

Using the BC at $r=R$ which is related to the surface charge density, we have

$$\sigma(\theta,\phi) = \varepsilon_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l+1)B_{lm}R^{-(l+2)}\,\mathrm{Y}_{lm}(\theta,\phi)$$

Using the orthogonality property of $\mathrm{Y}_{lm}(\theta,\phi)$, which states that

$$\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\,\mathrm{Y}_{lm}(\theta,\phi)Y^*_{l'm'}(\theta,\phi) = \delta_{mm'}\delta_{ll'}$$

we have

$$B_{lm} = \frac{R^{(l+2)}}{l+1}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\,\sigma(\theta,\phi)Y^*_{lm}(\theta,\phi)$$

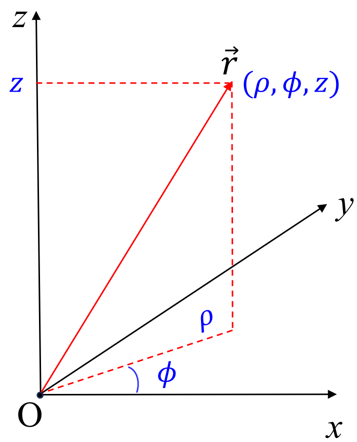


Fig. 7.5 The cylindrical coordinates.

7.5 Boundary Value Problems with Cylindrical Symmetry

In systems with cylindrical symmetry, the potential $\varphi(\rho, \phi, z)$ depends on the radius ρ, azimuthal angle ϕ, and height z along the cylinder's axis, as shown in **Figure 7.5**. When applying Laplace's equation in cylindrical coordinates, it is expressed as,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2}+\frac{\partial^2\varphi}{\partial z^2}=0. \tag{7.31}$$

To solve for the potential, we assume $\varphi(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z)$. Substituting this form into Laplace's equation and separating variables yields three ODEs,

$$\begin{cases} \dfrac{d^2Z}{dz^2}-k^2Z=0 \\ \dfrac{d^2\Phi}{d\phi^2}+v^2\Phi=0 \\ \dfrac{d^2R}{d\rho^2}+\dfrac{1}{\rho}\dfrac{dR}{d\rho}+\left(k^2-\dfrac{v^2}{\rho}\right)R=0 \end{cases}, \tag{7.32}$$

where k^2 and v^2 are separation constants that will affect the behavior and form of the solution. The general solutions for these three ODEs can be written as,

$$Z_k(z)=\begin{cases} A_0+B_0z, & k=0 \\ A_ke^{kz}+B_ke^{-kz}, & k\neq 0 \end{cases}, \tag{7.33}$$

$$\Phi_v(\phi)=\begin{cases} C_0+D_0\phi, & v=0 \\ C_ve^{iv\phi}+D_ve^{-iv\phi}, & v\neq 0 \end{cases}, \tag{7.34}$$

$$R(\rho)=\begin{cases}E_0+F_0\ln\rho, & k=0, v=0\\ E_v\rho^v+F_v\rho^{-v}, & k=0, v\neq 0\\ E_{kv}J_v(k\rho)+F_{kv}N_v(k\rho), & k^2>0\\ E_{kv}I_v(k\rho)+F_{kv}K_v(k\rho), & k^2<0\end{cases}. \tag{7.35}$$

where $J_v(x)$ and $N_v(x)$ are called Bessel functions and Neumann functions respectively, and $I_v(x)$ and $K_v(x)$ are modified Bessel functions of the first and second kinds. These functions have distinct behaviors, making them suitable for different boundary conditions and physical contexts

Bessel Functions $J_v(x)$

Bessel functions, named after Friedrich Bessel, are central to solving cylindrical and spherical problems in physics, such as vibrations of circular membranes and heat conduction in cylindrical objects. The Bessel functions of the first kind $J_v(x)$ are regular at $x = 0$, while the Neumann functions $N_v(x)$ (Bessel functions of the second kind) diverge at $x = 0$.

The Bessel functions $J_v(x)$ can be expressed in series,

$$J_v(x)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\,(v+i)!}\left(\frac{x}{2}\right)^{v+2i}$$

An alternative expression for $J_v(x)$ is its integral representation,

$$J_v(x)=\frac{1}{\pi}\int_0^{\pi}\cos(v\theta-x\sin\theta)d\theta$$

Its generating function for Bessel functions is

$$g(x,t)=e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)}=\sum_{v=-\infty}^{\infty}J_v(x)t^v$$

Bessel functions have the following properties:

1) $J_{-v}(x)=(-1)^v J_v(x)$
2) $J_{v+1}(x)=\frac{v}{x}J_v(x)-J'_v(x)$

For some lower orders, Bessel functions simplify to familiar expressions:

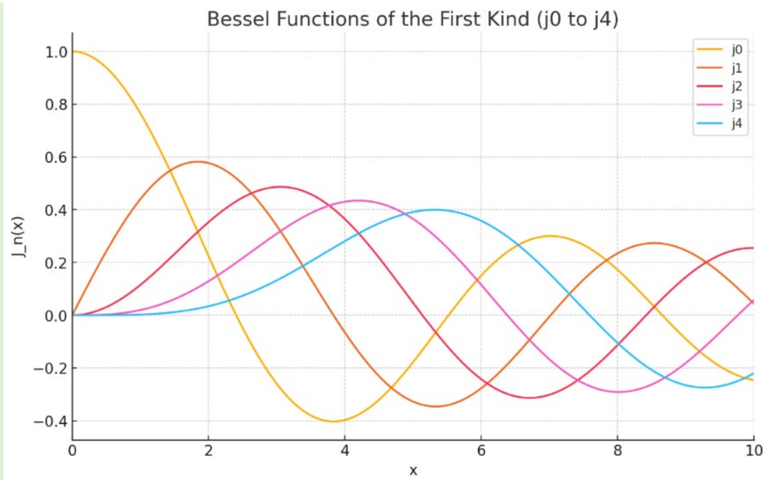
$$J_0(x)=\frac{\sin x}{x}$$

$$J_1(x)=\frac{\sin x}{x^2}-\frac{\cos x}{x}$$

$$J_2(x)=\left(\frac{3}{x^2}-1\right)\frac{\sin x}{x}-\frac{3\cos x}{x^2}$$

$$J_3(x)=\left(\frac{15}{x^3}-\frac{6}{x}\right)\frac{\sin x}{x}-\left(\frac{15}{x^2}-1\right)\frac{\cos x}{x}$$

Below is a representative plot of these Bessel functions:



$J_v(x)$ oscillates similarly to sine and cosine functions, with their amplitude gradually decaying as x increases, i.e., each Bessel function has multiple zero locations. Also, $J_v(x)$ is finite at $x = 0$.

Orthogonality: Let a_{vm} is the nth zero of $J_v(x)$, then

$$\int_0^a J_v\left(a_{vm}\frac{\rho}{a}\right) J_v\left(a_{vm'}\frac{\rho}{a}\right)\rho d\rho = \frac{a^2}{2}[J_{v+1}(a_{vm})]^2\delta_{mm'}$$

This orthogonality is useful in expanding functions in terms of Bessel functions when solving boundary value problems in cylindrical geometries.

Neumann function $N_v(x)$

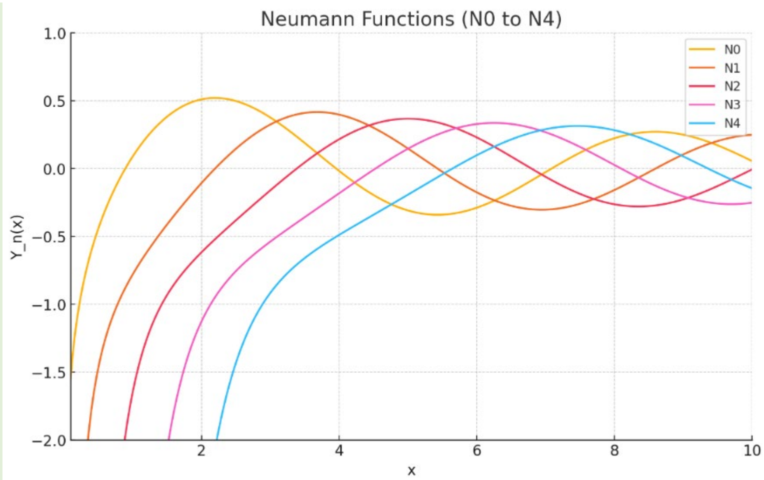
The Neumann functions (Bessel functions of the second kind) $N_v(x)$ can be written as,

$$N_v(x) = \frac{J_v(x)\cos v\pi - J_{-v}(x)}{\sin v\pi}$$

or,

$$N_v(x) = \frac{2}{\pi}\int_0^\pi \cos(x\cosh t)dt$$

Below are some plots of lowest order Neumann functions:



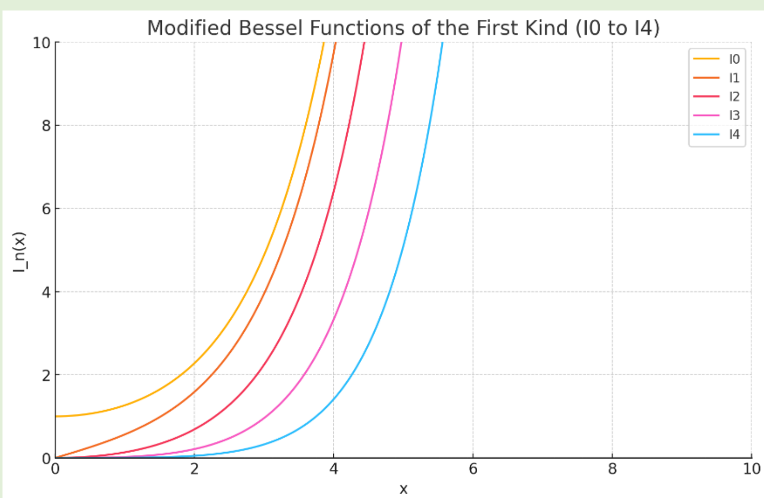
$N_v(x)$ also oscillates similarly to sine and cosine functions, with their amplitude gradually decaying as x increases. However, $J_v(x)$ approaches to $-\infty$ when $x \to 0$.

Modified Bessel Functions $I_v(x)$ and $K_v(x)$

The modified Bessel functions of the first kind $I_v(x)$ can be written as,

$$I_v(x) = e^{-\frac{iv\pi}{2}} J_v\left(xe^{\frac{i\pi}{2}}\right) = \sum_{i=0}^{\infty} \frac{1}{i!\,(v+i)!}\left(\frac{x}{2}\right)^{v+2i}$$

Below is a plot of some $I_v(x)$:

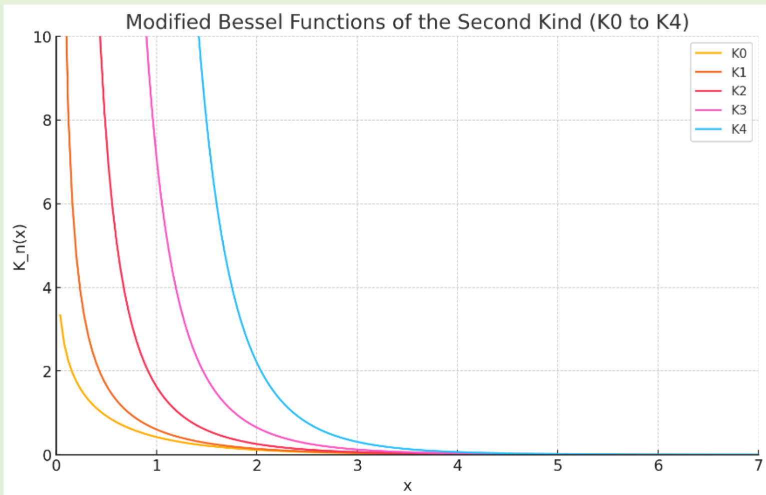


For small values of x, $I_v(x)$ behaves similarly to a power function, increasing gradually as x increases from zero. The function does not oscillate but grows smoothly from zero. The large-x approximation for $I_v(x)$ is $I_v(x) \approx \frac{e^x}{\sqrt{2\pi x}}$.

The modified Bessel functions of the second kind $K_v(x)$ can be expressed as,

$$K_v(x) = \frac{\pi}{2}\frac{I_{-v}(x) - I_v(x)}{\sin v\pi}$$

Below is a plot of some $K_v(x)$:



For small values of x, $K_v(x)$ diverges, meaning it goes to infinity as x approaches zero. The form of divergence depends on the order v,

$$K_v(x) \approx \begin{cases} -\ln x, & v = 0 \\ \frac{\Gamma(v)}{2}\left(\frac{2}{x}\right)^v, & v > 0 \end{cases}$$

The large-x approximation for $K_v(x)$ is

$$I_v(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}$$

Example 7.5 *Consider an infinitely long conducting cylinder of radius a, split along its long axis into two half-cylinders, with an infinitesimal gap separating them. A potential difference* V_0 *is applied between the two halves, such that one half has a potential of* $V_0/2$ *and the other has a potential of* $-V_0/2$*, as shown in* ***Fig. 7.6***. *Find the electrostatic potential inside the cylinder.*

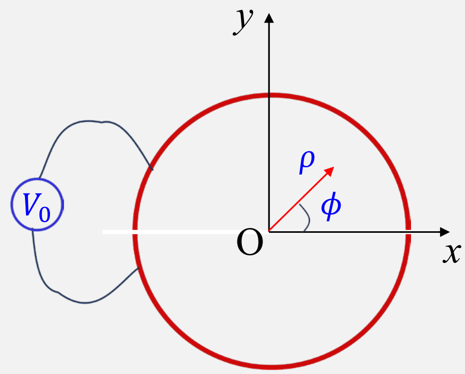


Fig. 7.6 A halved conducting cylinder with a potential difference applied across the two halves.

Discussion: Since the cylinder is assumed to be infinitely long, the problem can be treated as two-dimensional, with no dependence on the *z*-coordinate. Thus, the potential can be expressed as, $\varphi(\rho,\phi)$. Since there are no charges inside the cylinder, $\varphi(\rho,\phi)$ also satisfy the Laplace's equation. According to **Equations 7.33** and **7.35**, this implies that $k=0$. There are three BCs: in the center of the cylinder, $\varphi(\rho\to 0,\phi)$ shall be finite; to make the solution more symmetric, we can assume that the top cylinder has a potential of $V_0/2$, while the bottom cylinder has $-V_0/2$.

Solution:

Step 1. General Solution for $\varphi(\rho,\phi)$

Since $k=0$,according to **Equations 7.33 – 7.35**, the general solution for the $\varphi(\rho,\phi)$ can be written as,

$$\varphi(\rho,\phi)=\sum_{v=1}^{\infty}(C_v\cos v\phi+D_v\sin v\phi)(E_v\rho^v+F_v\rho^{-v})$$

The series starts from $\boldsymbol{v=1}$ because the v=0v = 0v=0 term would correspond to a constant potential, which is not relevant for this problem due to the specified boundary conditions.

Step 2. Implement of BCs

The BCs are

(1) At $\rho\to 0$, $\varphi(\rho\to 0,\phi)$ shall be finite

(2) $\varphi(a,\phi)=\begin{cases}\frac{V_0}{2} & \text{for } 0\le\phi<\pi\\ -\frac{V_0}{2} & \text{for } -\pi\le\phi<0\end{cases}$

To ensure that the potential remains finite at the origin, we must set $F_v = 0$ for all v. In addition, the potential changes sign between $V_0/2$ and $-V_0/2$, indicating that $\varphi(\rho, \phi)$ is an odd function in ϕ. Therefore, all C_v terms (cosine terms) must be zero, leaving,

$$\varphi(\rho, \phi) = \sum_{v=1}^{\infty} D_v \sin v\phi \, \rho^v$$

Step 3. Determination of D_v
To find the coefficients D_v, we apply the boundary condition at $\rho = a$,

$$\varphi(a, \phi) = \sum_{v=1}^{\infty} D_v a^v \sin v\phi$$

We utilize the orthogonality of the sine functions to solve for D_v. Multiplying both sides of the boundary equation by $\sin v'\phi$ and integrating from $-\pi$ to π gives

$$\int_{-\pi}^{\pi} \varphi(a, \phi) \sin v'\phi \, d\phi = \int_{-\pi}^{\pi} \sum_{v=1}^{\infty} D_v a^v \sin v\phi \sin v'\phi \, d\phi$$

The left-hand side integral evaluates to

$$\int_{-\pi}^{\pi} \varphi(a, \phi) \sin v'\phi \, d\phi = \int_{0}^{\pi} \frac{V_0}{2} \sin v'\phi \, d\phi - \int_{-\pi}^{0} \frac{V_0}{2} \sin v'\phi \, d\phi$$

$$= \frac{V_0}{v'} \left[1 - (-1)^{v'}\right]$$

The right-hand side becomes

$$\int_{-\pi}^{\pi} \sum_{v=1}^{\infty} D_v a^v \sin v\phi \sin v'\phi \, d\phi = \int_{-\pi}^{\pi} D_v a^v \sum_{v=1}^{\infty} \sin v\phi \sin v'\phi \, d\phi = \pi D_v a^v$$

Compare the right and left sides, we obtain

$$D_v = \frac{V_0}{\pi v a^v} [1 - (-1)^v]$$

Step 4. Final solution for the potential
The final expression for the electrostatic potential inside the cylinder is

$$\varphi(a, \phi) = \sum_{v=1}^{\infty} \varphi(a, \phi) = \sum_{v=1}^{\infty} \frac{V_0}{\pi v a^v} [1 - (-1)^v] \, \rho^v \sin v\phi$$

The term $[1 - (-1)^v]$ ensures that only odd harmonics contribute to the solution, reflecting the odd symmetry of the potential with respect to φ. Also, the coefficients D_v decrease as v increases, indicating that higher harmonics contribute less to the overall potential.

➢ ***Special Cases of Boundary Value Problems with Cylindrical Symmetry***

In cylindrical symmetry, Laplace's equation can often be simplified under specific boundary conditions and assumptions. Below, we outline four typical cases where symmetry allows us to reduce the problem to two-dimensional or simplified forms, each leading to particular solution structures.

(1) Infinite *z*-direction with boundary conditions in ρ and ϕ
For a cylinder that extends infinitely along the *z*-axis with boundary conditions only in the radial and azimuthal directions, the potential $\varphi(\rho, \phi)$ does not depend on z. Thus, Laplace's equation, **Equation 7.31**, reduces to,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \varphi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \varphi}{\partial \phi^2} = 0. \tag{7.36}$$

This implies that $k = 0$, and the solutions for the functions of ρ and ϕ are,

$$\Phi_v(\phi) = \begin{cases} C_0 + D_0\phi, & v = 0 \\ C_v e^{iv\phi} + D_v e^{-iv\phi}, & v \neq 0 \end{cases}, \tag{7.37}$$

$$\mathrm{R}_v(\rho) = \begin{cases} E_0 + F_0 \ln\rho, & v = 0 \\ E_v \rho^v + F_v \rho^{-v}, & v \neq 0 \end{cases}, \tag{7.38}$$

The general solution $\varphi(\rho, \phi)$ inside the cylinder can be expressed as,

$$\varphi(\rho, \phi) = (C_0 + D_0\phi)(E_0 + F_0 \ln\rho) + \sum_{v=1}^{\infty}(C_v \cos v\phi + D_v \sin v\phi)(E_v\rho^v + F_v\rho^{-v}) \tag{7.39}$$

(2) Semi-infinite cylinder
If the cylinder has azimuthal symmetry and extends infinitely in the positive or negative *z*-direction, the potential $\varphi(\rho, z)$ is independent of ϕ, reducing Laplace's equation to,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \varphi}{\partial \rho}\right) + \frac{\partial^2 \varphi}{\partial z^2} = 0. \tag{7.40}$$

In this case, we impose $\varphi(\rho, z \to \infty) \to 0$ as $|z| \to \infty$, indicating the solution $Z_k(z)$ has the form

$$Z_k(z) = \begin{cases} A_0 + B_0 z, & k = 0 \\ A_k e^{kz} + B_k e^{-kz}, & k \neq 0 \end{cases}, \tag{7.41}$$

with $k^2 > 0$. For $R_v(\rho)$, we use Bessel functions, leading to

$$R_v(\rho) = \begin{cases} E_0 + F_0 \ln\rho, & k = 0 \\ E_{kv} J_v(k\rho) + F_{kv} N_v(k\rho), & k^2 > 0 \end{cases}, \tag{7.42}$$

Since φ is finite at $\rho = 0$, we set $F_{kv} = 0$, giving

$$R_v(\rho) = E_{kv} J_v(k\rho) \tag{7.43}$$

Thus, the general solution $\varphi(\rho, z)$ inside the cylinder can be expressed as,

$$\varphi(\rho, z) = A_0 + \sum_{k=1}^{\infty}\left(A_k e^{kz} + B_k e^{-kz}\right) J_v(k\rho). \tag{7.44}$$

(3) Cylinders with two lids

For a finite cylinder with closed top and bottom surfaces, assuming azimuthal symmetry, Laplace's equation also reduces to **Equation 7.40**. However, due to the finite height h with top and bottom surfaces at specified potentials, the solution in z takes the form,

$$Z_k(z) = \begin{cases} A_0 + B_0 z, & k = 0 \\ A_k \cos kz + B_k \sin kz, & k \neq 0 \end{cases}, \tag{7.45}$$

where k is chosen such that $k^2 < 0$ to satisfy boundary conditions on the top and bottom surfaces. Thus,

$$R_v(\rho) = E_{kv} I_v(k\rho) + F_{kv} K_v(k\rho). \tag{7.46}$$

The general solution for $\varphi(\rho, z)$ within the cylinder then becomes,

$$\varphi(\rho, z) = \sum_{k=1}^{\infty} (A_k \cos kz + B_k \sin kz)[E_{kv} I_v(k\rho) + F_{kv} K_v(k\rho)]. \tag{7.47}$$

(4) <u>Cylinder with boundary condition $\varphi(\rho = a, \phi, z) = 0$</u>

When the boundary at $\rho = a$ is set to zero potential, the solution can incorporate Bessel functions that satisfy the boundary conditions. In this case, we express $\varphi(\rho, \phi, z)$ as

$$\varphi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{v=1}^{\infty} (C_{vm} \cos k_{vm}\phi + D_{vm} \sin k_{vm}\phi) J_v(k_{vm}\rho) \sinh(k_{vm} z), \tag{7.48}$$

where $k_{vm} = a_{vm}/2$, and a_{vm} is the m-th zero of $J_v(x)$. This ensures that $\varphi(\rho = a, \phi, z) = 0$ at $\rho = a$, satisfying the boundary condition by using the zeros of the Bessel functions $J_v(x)$.

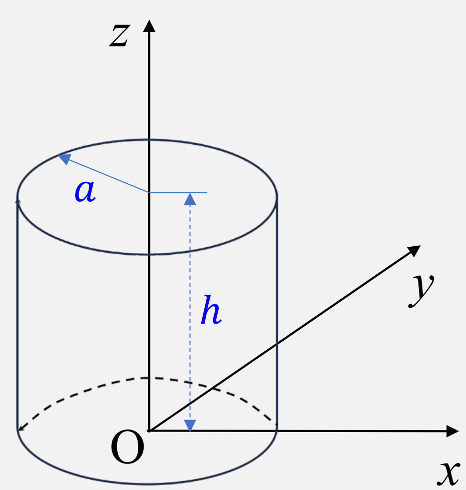


Fig. 7.7 A cylinder with zero potential on its top and bottom surfaces and a fixed potential on its side.

Example 7.6 *Consider a cylinder of height h and radius a, with its top and bottom surfaces held at zero potential, while the cylindrical side surface is maintained at a fixed potential V_0, as shown in **Fig.** 7.7. Find the electrostatic potential inside the cylinder.*

Discussion: Since the potential is fixed on the side of the cylinder and is independent of the azimuthal angle, we conclude that the potential φ does not depend on the angle ϕ, and thus we can express it as $\varphi(\rho, z)$. This implies that we only need to solve for radial (ρ) and vertical (z) components, with $v = 0$ in the general solution for cylindrical coordinates (**Equation 7.34**). Given the boundary conditions where the top and bottom surfaces are held at zero potential, the solution in the z-direction must be either a sine or cosine function, requiring $k^2 < 0$ to ensure periodicity in *z*. Consequently, the radial function $R(\rho)$ should be a combination of modified Bessel functions $I_0(k\rho)$ and $K_0(k\rho)$, as these functions naturally arise in cylindrical systems without angular dependence.

Solution: Based on the above reasoning, the general form for the potential $\varphi(\rho, z)$ within the cylinder is,

$$\varphi(\rho, z) = \sum_{k=1}^{\infty} (A_k \cos kz + B_k \sin kz)\,[E_k I_0(k\rho) + F_k K_0(k\rho)]$$

The corresponding BCs are

(1) $\varphi(\rho, 0) = 0$
(2) $\varphi(\rho, h) = 0$
(3) $\varphi(a, z) = V_0$
(4) When $\rho \to 0$, $\varphi(\rho, z)$ shall be finite

Step 1. Applying the finite potential condition (BC #4)

Since $K_0(k\rho) \to \infty$ as $\rho \to 0$, it is not physically valid for potential in $\rho < a$ region, i.e., $F_k = 0$ for all k. Therefore, the potential can be written as,

$$\varphi(\rho, z) = \sum_{k=1}^{\infty} (A_k \cos kz + B_k \sin kz)\, I_0(k\rho)$$

Implement BC#1 and BC#2, we have

$$\begin{cases} A_k \cos 0 + B_k \sin 0 = 0 \\ A_k \cos kh + B_k \sin kh = 0 \end{cases} \Rightarrow \begin{cases} A_k = 0 \\ B_k \neq 0 \end{cases}$$

In order to have the second equation equals to zero, we have,

$$k = \frac{n\pi}{h}, \qquad n = 1, 2, \cdots$$

Thus, the potential reduces to

$$\varphi(\rho, z) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{h} z\right) I_0\left(\frac{n\pi}{h}\rho\right)$$

To find out B_n, we can implement BC#3,

$$\varphi(a, z) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{h} z\right) I_0\left(\frac{n\pi}{h}a\right)$$

Use the orthogonality property of sine functions, multiply $\sin\left(\frac{n'\pi}{h} z\right)$ on both sides and perform the integration with respect to *z*, we have

$$\int_0^h V_0 \sin\left(\frac{n'\pi}{h} z\right) dz = \int_0^h \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{h} z\right) I_0\left(\frac{n\pi}{h}a\right) \sin\left(\frac{n'\pi}{h} z\right) dz$$

We can obtain

$$B_{n'} = \frac{2V_0}{n'\pi I_0\left(\frac{n'\pi}{h}a\right)}\left[1 - (-1)^{n'}\right]$$

Therefore,

$$\varphi(\rho, z) = \sum_{n=1}^{\infty} \frac{2V_0}{n\pi I_0\left(\frac{n\pi}{h}a\right)}\left[1 - (-1)^{n}\right] \sin\left(\frac{n\pi}{h} z\right) I_0\left(\frac{n\pi}{h}\rho\right)$$